Superposition of homogeneous strain and progressive deformation in rocks

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The geometry of strain superposition in two and three dimensions is treated mathematically. The first half of the paper is devoted to the two-dimensional case. After an introductory study of examples on sequential superposition of pure- and simple shear simultaneous superposition of these two classes of strain is treated more thoroughly. Simultaneous superposition implies progressive deformation. In the course of progressive deformation the finite strain is determined by the integrated version of the rate-of-deformation equation. Integration of the rate-of-deformation equation yields the particle-path equation which describes the path of any particle in the deforming body. Applied on the set of particles lying on a circle in the undeformed body the particle-path equation gives the strain ellipse and describes the progression of the strain ellipse in time. In general the particle paths are open curved lines. Special combinations of simple shear and pure shear give, however, closed particle paths which constitute sets of concentric ellipses. Under such circumstances the strain ellipse pulzates and rotates completely around the clock during the deformation, the number of rotations depending only on the extent of final strain.

The second half of the paper treats the geometry of three-dimensional strain. Also in this part examples on sequential superposition of two classes of strain are firstly considered as introduction to the more interesting simultaneous superposition. Rate-of-deformation equations for the three-dimensional simultaneous superposition of strain are developed. These are integrated to form the particle-path equations in three dimensions. From the latter the finite strain, and in particular the strain ellipsoid, follow at any moment which we choose to consider during the deformation. For special combinations of irrotational three-dimensional strain and simple shear in a direction inclined to the principal axes for the irrotational strain the particle paths assume the form of three-dimensional spirals. The corresponding strain ellipsoid undergoes a kind of pulzating motion at the same time as it deforms progressively. For example, the long principal axis may continue to grow with time while the short and the median axes pulzate, their product, however, decreasing continuously with time. (The latter condition follows from the restriction of the theory to incompressible substances — i.e. the volume of the strain ellipsoid remains constant.)

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The two-dimensional case

Introduction

The two chief classes of two-dimensional strain of incompressible materials are the irrotational pure shear and the rotational simple shear. These are, however, special cases of strain which are applicable to rocks. Indeed, the strain in rocks is generally three-dimensional; if one nevertheless, however, wishes to simplify the treatment by assuming a two-dimensional geometry one at least must consider a more complicated deformation geometry than that defined either by pure shear or by simple shear. Neglecting the elastic compressibility of rocks and assuming chemically closed systems (no material transport to and fro the volume of rock under consideration) we shall find that a combination of pure shear and simple shear is general enough to describe most kinds of plane strain which rocks may undergo. The combination may either be simultaneous or sequential.

Pure shear is an irrotational finite plane strain defined by compression in one direction (the principal compressive strain) and a volume-conserving extension (principal extensive strain) normal to the compression. Both longitudinal strain and shear strain vanish in the direction normal to the plane containing the two principal axes of strain. The paths of movement of the particles are families of hyperbolas whose axes bisect the angle between the axes of principal strains. In the following we shall let the principal extensive strain coincide with the x axis of our orthogonal coordinate system and the principal compressive strain coincide with the y axis.

In this coordinate system finite pure shear is described by the linear transformation

$$(1)\qquad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (1+\varepsilon_x) & 0 \\ 0 & (1+\varepsilon_y) \end{bmatrix} \begin{bmatrix} x_o \\ y_o \end{bmatrix}.$$

Here x_o and y_o are the initial coordinates to a particle which is being displaced with the body, and x and y the coordinates to the same particle after the deformation. ε_x and ε_y are the finite principle strains in the directions x and y respectively.

Simple shear is a rotational plane strain defined by a finite shear strain in a given direction which remains fixed (does not rotate) in the deforming body and along which longitudinal strain vanishes. This direction may be distinguished as the simple shear direction and should not be confused with the direction of maximum shear strain. The particle paths are straight lines parallel to the simple shear direction.

Let the simple shear direction coincide with the axis x' in our coordinate system, y' being normal to x'. Simple shear is then described by the linear transformation

$$(2)\qquad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x'_o \\ y'_o \end{bmatrix},$$

where γ defines the magnitude of finite simple shear strain. x'_o, y'_o are the initial coordinates to a particle which is being displaced with the body and x', y' the final coordinates to the same particle.

Sequential superposition of pure and simple shear

Pure and simple shear are rather artificial types of strain which, however, may be combined to give more realistic strain patterns of the kinds sometimes occurring in deformed rocks. The composite finite strain may either be the result of sequential superposition or of simultaneous superposition. It is known that the composite finite strain resulting from the sequential superposition of two or more less complex strain geometries generally depends upon the order of superposition; see for example Ramsay, 1967. The treatment of an example may be informative. To make the superposition as general as possible without, however, departing from two-dimensional geometry, we let the direction of simple shear (the x' axis)

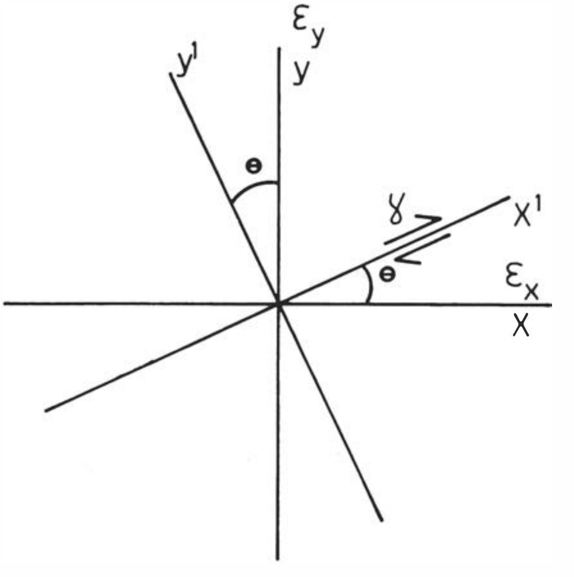


Fig. 1. Relative orientation of the coordinate system x, y for pure shear and the system x', y' for simple shear.

in the simple-shear deformation deviate by an angle θ in the anti-clockwise sense from the axis of principal extensive strain in the pure-shear deformation (Fig. 1). It is convenient to select the axes of principal strain in the pure-shear deformation as the coordinate axes to which all displacements will be referred. The x axis coincides with the strain ε_x and the y axis parallels the strain ε_y. The displacements associated with the pure shear are then given by the linear transformation already considered (eq. (1)). To make the simple-shear displacements relate to the same coordinate system we must rotate the coordinate axes x' and y' (eq. (2)) clockwise through an angle θ in relation to the simple shear direction. This rotation corresponds to a transformation of coordinates between the two coordinate system thus

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \tag{3}$$

where x', y' are the coordinates in the system whose x' axis parallels the simple-shear direction and x, y are the coordinates to the same point in space in a system whose x axis is rotated clockwise through the angle θ relative to the axis x'.

Transformation (3) applies both to the initial coordinates and to the final coordinates in the simple-shear equation (2) which consequently takes the form presented as eq. (4).

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_o \\ y_o \end{bmatrix}. \tag{4}$$

Expansion of eq. (4) yields

(5)

(a) $x\cos\theta + y\sin\theta = (\cos\theta - \gamma\sin\theta)x_o + (\sin\theta + \gamma\cos\theta)y_o$

(b) $-x\sin\theta + y\cos\theta = -x_o\sin\theta + y_o\cos\theta.$

Solved for x and y these equations give

(6)

(a) $x = (1 - \gamma\sin\theta\cos\theta)x_o + \gamma(\cos^2\theta)y_o$

(b) $y = -(\gamma\sin^2\theta)x_o + (1 + \gamma\sin\theta\cos\theta)y_o,$

or expressed in matrix form:

)
$$\begin{bmatrix} x_\gamma \\ y_\gamma \end{bmatrix} = \begin{bmatrix} (1-\gamma\sin\theta\cos\theta) & \gamma\cos^2\theta \\ -\gamma\sin^2\theta & (1+\gamma\cos\theta\sin\theta) \end{bmatrix} \begin{bmatrix} x_{o\gamma} \\ y_{o\gamma} \end{bmatrix}.$$

Equation (7) transforms particles from x_o, y_o to x, y when the simple shear strain is γ and the simple shear direction makes an angle θ with the x axis.

In the same coordinate system the pure shear corresponds to the transformation shown as eq. (8).

$$\begin{bmatrix} x_\varepsilon \\ y_\varepsilon \end{bmatrix} = \begin{bmatrix} (1+\varepsilon_x) & 0 \\ 0 & (1+\varepsilon_y) \end{bmatrix} \begin{bmatrix} x_{o\varepsilon} \\ y_{o\varepsilon} \end{bmatrix}. \tag{8}$$

Assume now that a substance (a rock) is first deformed in pure shear and subsequently in simple shear. In this case the final coordinates x_ε and y_ε after the pure-shear deformation are to be taken as the initial coordinates $x_{o\gamma}$ and $y_{o\gamma}$ for the subsequent simple shear deformation. Hence expression (9) follows

$$\begin{bmatrix} x_{o\gamma} \\ y_{o\gamma} \end{bmatrix} = \begin{bmatrix} x_\varepsilon \\ y_\varepsilon \end{bmatrix} = \begin{bmatrix} (1+\varepsilon_x) & 0 \\ 0 & (1+\varepsilon_y) \end{bmatrix} \begin{bmatrix} x_{o\varepsilon} \\ y_{o\varepsilon} \end{bmatrix}, \tag{9}$$

which must be inserted in eq. (7) to obtain the composite transformation which corresponds to the sequence pure shear overprinted by simple shear. The composite transformation takes the form shown in eq. (10).

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (1-\gamma\sin\theta\cos\theta) & \gamma\cos^2\theta \\ -\gamma\sin^2\theta & (1+\gamma\sin\theta\cos\theta) \end{bmatrix} \begin{bmatrix} (1+\varepsilon_x) & 0 \\ 0 & (1+\varepsilon_y) \end{bmatrix} \begin{bmatrix} x_o \\ y_o \end{bmatrix}. \tag{10}$$

Carrying out the matrix mulitplication we obtain

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (1-\gamma\sin\theta\cos\theta)(1+\varepsilon_x) & \gamma\cos^2\theta(1+\varepsilon_y) \\ -\gamma\sin^2\theta(1+\varepsilon_x) & (1+\gamma\sin\theta\cos\theta)(1+\varepsilon_y) \end{bmatrix} \begin{bmatrix} x_o \\ y_o \end{bmatrix}. \tag{11}$$

On the other hand we may reverse the order, starting with simple shear and following up with pure shear. In this sequence we have

$$\begin{bmatrix} x_{o\varepsilon} \\ y_{o\varepsilon} \end{bmatrix} = \begin{bmatrix} x_\gamma \\ y_\gamma \end{bmatrix} \tag{12}$$

and the composite transformation becomes

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (1+\varepsilon_x) & 0 \\ 0 & (1+\varepsilon_y) \end{bmatrix} \begin{bmatrix} (1-\gamma\sin\theta\cos\theta) & \gamma\cos^2\theta \\ -\gamma\sin^2\theta & (1+\gamma\sin\theta\cos\theta) \end{bmatrix} \begin{bmatrix} x_o \\ y_o \end{bmatrix} \tag{13}$$

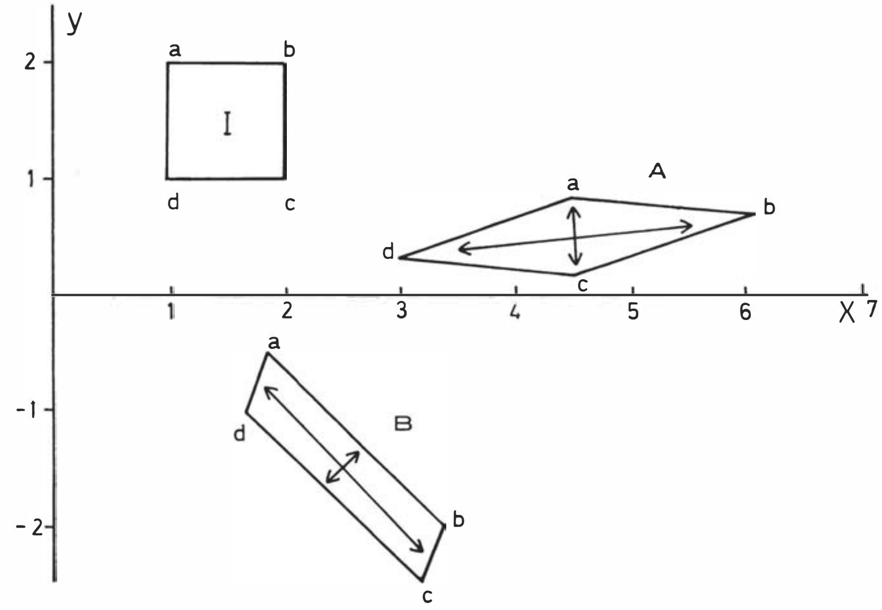


Fig. 2. An initial square (I) deformed in sequential superposition of simple shear and pure shear. The axes x and y coincide with the principle strains for the pure shear while the simple shear direction is inclined to the coordinate axes, see text. A shows the result when simple shear precedes pure shear, B when the order of superposition is reversed. The crossed arrows define the principal axes of strain. The origin of the coordinate system is fixed in the deforming body.

or

$$(14)\qquad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (1-\gamma\sin\theta\cos\theta)(1+\varepsilon_x) & \gamma\cos^2\theta(1+\varepsilon_x) \\ -\gamma\sin^2\theta(1+\varepsilon_y) & (1+\gamma\sin\theta\cos\theta)(1+\varepsilon_y) \end{bmatrix} \begin{bmatrix} x_o \\ y_o \end{bmatrix}.$$

Since the square matrix of eq. (14) is not identical to the square matrix of eq. (11) the final coordinates x, y will generally not be identical in the two sequential superpositions of opposite order if the initial coordinates are the same.

As an example assume $\varepsilon_x = 2$, $\varepsilon_y = -\frac{2}{3}$, $\gamma = 1$, and $\theta = 45°$. Inserted in eq. (11) these numerical values yield eq. (15)

$$(15)\qquad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1{,}5000 & 0{,}16667 \\ -1{,}5000 & 0{,}5000 \end{bmatrix} \begin{bmatrix} x_o \\ y_o \end{bmatrix},$$

which gives the final transformation of points in the case that pure shear precedes simple shear. The deformation of a square is shown in Fig. 2B.

In the case that simple shear precedes pure shear the numerical values in our example must be inserted in eq. (14). The result is eq. (16).

$$(16)\qquad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1{,}5000 & 1{,}5000 \\ -0{,}16667 & 0{,}5000 \end{bmatrix} \begin{bmatrix} x_o \\ y_o \end{bmatrix}.$$

The deformation of a square according to this transformation is shown in Fig. 2A.

In the special cases that the direction of simple shear coincides with the x- or the y axis the angle θ is 0° and 90° respectively. For $\theta = 0$ the transformation for the superposition takes the form shown in eq. (17) in the case that pure shear precedes simple shear, and the form shown in eq. (18) if the sequence is reversed.

$$(17)\qquad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (1+\varepsilon_x) & \gamma(1+\varepsilon_y) \\ 0 & (1+\varepsilon_y) \end{bmatrix} \begin{bmatrix} x_o \\ y_o \end{bmatrix}$$

$$(18)\qquad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (1+\varepsilon_x) & \gamma(1+\varepsilon_x) \\ 0 & (1+\varepsilon_y) \end{bmatrix} \begin{bmatrix} x_o \\ y_o \end{bmatrix}$$

Since generally $\varepsilon_x \gtrless \varepsilon_y$ the coefficient matrices of eqs. (17) and (18) are not identical and it follows that the resulting composite strain also in this case depends upon the order in which the two types of deformation are superimposed.

The deformation of an initial circle into the finite strain ellipse probably gives more information on the various details of the strain geometry than the deformation of any other kind of figure. We shall therefore study the strain ellipses created by the sequential superposition. The procedure to follow is firstly to solve the appropriate equation for the finite deformation (i.e. one of the eqs. (11), (14), (17) or (18)) with respect to the initial coordinates x_o and y_o. The expressions for x_o and y_o thus obtained are then inserted in the equation for the initial circle, eq. (21). The result is the equation for the strain ellipse.

Suppose that we write the general equation for the composite deformation as follows (eq. 19)):

$$(19)\qquad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} A & B \\ D & E \end{bmatrix} \begin{bmatrix} x_o \\ y_o \end{bmatrix},$$

where the coefficients A, B, D and E depend upon which superposition we are studying (i.e. whether it is eq. (11), (14), (17) or (18)).

Inversion of the matrix gives the initial coordinates expressed in terms of the final coordinates, thus:

$$(20)\qquad \begin{bmatrix} x_o \\ y_o \end{bmatrix} = \begin{bmatrix} \dfrac{E}{\triangle} & -\dfrac{B}{\triangle} \\ -\dfrac{D}{\triangle} & \dfrac{A}{\triangle} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Here $\triangle = AE - BD$ is the determinant to the coefficient matrix of eq. (19).

The initial circle with unit radius is described by eq. (21):

$$(21)\qquad x_o^2 + y_o^2 = 1,$$

in which we insert the expressions for x_o and y_o in order to obtain the deformed circle, i.e. the strain ellipse, eq. (22),

$$(22)\qquad \frac{E^2+D^2}{(AE-BD)^2}x^2 - 2\frac{BE+AD}{(AE-BD)^2}xy + \frac{A^2+B^2}{(AE-BD)^2}y^2 = 1.$$

This is the equation for an ellipse (or another conic section depending upon the character of the coefficients) with center in the origin and axes generally inclined to the coordinate axes.

One way of determining the axes of the strain ellipse and its inclination is to put the equation for the ellipse in matrix form, thus:

$$(23)\qquad (x\ y)\begin{bmatrix} \dfrac{E^2+D^2}{\triangle^2} & -\dfrac{BE+AD}{\triangle^2} \\ -\dfrac{BE+AD}{\triangle^2} & \dfrac{A^2+B^2}{\triangle^2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 1$$

and to take the eigenvalue of the coefficient matrix of the ellipse equation. (For the method of determining the eigenvalues of a matrix see e.g. Hadley, 1961.) As discussed for example by Hammarling (1970 p. 20), the two eigenvalues of the 2×2 matrix of the strain ellipse equation furnish both the length of the axes, say r_1 and r_2, and the slope of the axes, say y_1/x_1 and y_2/x_2. The relationships are

$$(24)\qquad \text{(a)}\quad r_1 = \frac{1}{\sqrt{\lambda_1}}, \qquad \text{(b)}\quad r_2 = \frac{1}{\sqrt{\lambda_2}},$$

$$(25)\qquad \text{(a)}\quad \mathrm{tg}\Phi_1 = y_1/x_1 = \frac{E^2+D^2-\lambda_1}{BE+AD} = \frac{BE+AD}{A^2+B^2-\lambda_1},$$

$$\text{(b)}\quad \mathrm{tg}\Phi_2 = y_2/x_2 = \frac{E^2+D^2-\lambda_2}{BE+AD} = \frac{BE+AD}{A^2+B^2-\lambda_2}.$$

Here λ_1 and λ_2 are the two eigenvalues of the coefficient matrix in eq. (23). The eigenvalues as expressed in terms of the coefficients in the matrix are

$$(26)\qquad \lambda_i = \frac{1}{2(AE-BD)^2}\left(A^2+B^2+D^2+E^2 \pm \pm\sqrt{(A^2+B^2-D^2-E^2)^2+4(BE+AD)^2}\right)$$

The axes of the strain ellipses for our two examples of sequential superposition are shown in Figs. 2A and 2B.

Simultaneous superposition of strain: progressive deformation

A plastic or viscous body may be strained in a fashion that can be treated as a simultaneous superposition of two or more classes of less complex deformation, such as e.g. pure shear and simple shear. When this occurs the composite strain at any moment during the deformation depends upon the absolute and the relative rate of change of the pure shear and of the simple shear.

The rate of change of longitudinal strain is commonly identified by the symbol $\dot{\varepsilon} \equiv \frac{d\varepsilon}{dt}$ and the rate of change of shear strain by the symbol $\dot{\gamma} \equiv \frac{d\gamma}{dt}$ where t is time.

Note that we define $d\varepsilon$ as dl/l (i.e. infinitesimal natural strain) and not as dl/l_0. The difference between dl/l and dl/l_0 is that l is the distance between two points at any instance we wish to consider during the straining process while l_0 is the distance between the points prior to straining. The symbol $\dot{\varepsilon}$ is accordingly identical to $\frac{dl/dt}{l}$ where dl/dt is the rate of displacement.

In the above discussion of sequential superposition we were interested in the finite strain and the corresponding finite transformation of points such as defined by eqs. (11), (14), (17) and (18). These equations inform only on the relation between the initial geometry and the final geometry. No information is supplied on the path from the initial- to the final state. We are now interested in the progression of the deformation from the initial- to the final situation. That means we must consider the rate of change of strain and the corresponding rate of change of displacement of points. In other words we must consider the differential equations which relate displacement to strain. The rates of change of the displacement components x and y to a point in a *homogeneously* deforming body are related to the rate of change of strain according to the simple eqs. (27) and (28). "Homogeneously deforming" means that the rate of change of strain is constant throughout the body.

$$(27) \qquad \begin{bmatrix} \dot{x}_\varepsilon \\ \dot{y}_\varepsilon \end{bmatrix} = \begin{bmatrix} \dot{\varepsilon}_x & 0 \\ 0 & \dot{\varepsilon}_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

is valid for pure shear, and

$$(28) \qquad \begin{bmatrix} \dot{x}_\gamma \\ \dot{y}_\gamma \end{bmatrix} = \begin{bmatrix} 0 & \dot{\gamma} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

for simple shear when the simple-shear direction coincides with the x axis.

In the general case when the simple-shear direction makes an angle θ with the x axis the equation becomes somewhat more complicated. The sought equation can be shown to be:

$$(29) \qquad \begin{bmatrix} \dot{x}_\gamma \\ \dot{y}_\gamma \end{bmatrix} = \begin{bmatrix} -\dot{\gamma}\sin\theta\cos\theta & \dot{\gamma}\cos^2\theta \\ -\dot{\gamma}\sin^2\theta & \dot{\gamma}\sin\theta\cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

In the above equations $\dot{x} \equiv \frac{dx}{dt}$ and $\dot{y} \equiv \frac{dy}{dt}$.

The subscripts ε and γ indicate whether the displacements relate to pure shear or to simple shear.

Equation (29) follows when we operate on either side of eq. (28) with the rotation matrix $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ similarly as done in eq. (4).

In the course of the simultaneous superposition of the two classes of strain any particle in the body moves along a path determined by the combined geometry of the pure shear and the simple shear. Therefore, at any moment during the deformation the coordinates x, y to a particle are to be used simultaneously both in the equation for pure shear and in the equation for simple shear. In other words, eq. (27) and eq. (29) (we choose to use the general simple-shear transformation) are simultaneous equations when the two strains are combined contemporaneously. This means that the simultaneous superposition of the two kinds of strain is accomplished mathematically by adding the coefficient matrices in eqs. (27) and (29); at the same time the column matrices on the left side of the equations must also be added. (Note that the adding of the coefficient matrices in the present case of simultaneous superposition contrasts the multiplication of the coefficient matrices in the above case of sequential superposition.) The adding yields

$$(30) \qquad \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \dot{\varepsilon}_x - \dot{\gamma}\sin\theta\cos\theta & \dot{\gamma}\cos^2\theta \\ -\dot{\gamma}\sin^2\theta & \dot{\varepsilon}_y + \dot{\gamma}\sin\theta\cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Here $\dot{x} = \dot{x}_\varepsilon + \dot{x}_\gamma$ and $\dot{y} = \dot{y}_\varepsilon + \dot{y}_\gamma$.

If the rates of change of strain, $\dot{\varepsilon}$, $\dot{\gamma}$, are kept constant during the deformation and likewise the direction of simple shear, θ, is fixed eq. (30) constitutes a system of ordinary linear differential equations with constant coefficients inasmuch as $\dot{x} \equiv \frac{dx}{dt}$ and $\dot{y} \equiv \frac{dy}{dt}$. We shall refer to eq. (30) as the *rate-of-displacement equation*.

Two real and distinct eigenvalues of the matrix of the rate-of-displacement equation. — The solution of the system of differential equations (30) takes different forms depending upon whether the two eigenvalues of the coefficient matrix are distinct, coincident, zero, real or complex. If the eigenvalues are real and distinct the general form of the solution is:

$$(31) \qquad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} \exp(\varkappa_1 t) \\ \exp(\varkappa_2 t) \end{bmatrix}.$$

(See books on ordinary differential equations, e.g. Kreider et al., 1968.) Here $\varkappa_1$ and $\varkappa_2$ are the eigenvalues to the coefficient matrix in eq. (30)

and c_{ij} are constants determined partly by the initial coordinates, x_o, y_o, to the particle whose path we wish to follow, partly by the magnitudes of $\dot{\varepsilon}$, $\dot{\gamma}$ and θ. The eigenvalues are also determined by the magnitudes of $\dot{\varepsilon}$, $\dot{\gamma}$ and θ such that all quantities except the time, t, are constant in eq. (31) if we have selected the initial position of a particle and $\dot{\varepsilon}$, $\dot{\gamma}$ and θ do not vary during the deformation. Equation (31) consequently shows how the coordinates to a particle change with time, i.e. the equation describes the particle path. We shall sometime refer to eq. (31) as the *particle-path equation.*

For the further discussion we simplify the notation of the coefficient matrix in eq. (30), thus:

$$(32)\quad \begin{bmatrix} \dot{\varepsilon}_x - \dot{\gamma}\sin\theta\cos\theta & \dot{\gamma}\cos^2\theta \\ -\dot{\gamma}\sin^2\theta & \dot{\varepsilon}_y + \dot{\gamma}\sin\theta\cos\theta \end{bmatrix} \equiv \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

The eigenvalues follow then from the determinant equation

$$(33)\quad \begin{vmatrix} (a_{11} - \varkappa) & a_{12} \\ a_{21} & (a_{22} - \varkappa) \end{vmatrix} = 0$$

which gives

$$(34)\quad \varkappa_i = \tfrac{1}{2}(a_{11} + a_{22}) \pm \pm \tfrac{1}{2}\sqrt{(a_{11} - a_{22})^2 + 4a_{12}a_{21}},$$

where the positive and the negative square root associate with respective $\varkappa_1$ and $\varkappa_2$.

Having thus determined $\varkappa_1$ and $\varkappa_2$ — which are to go in the exponents in the particle-path equation — expressed in terms of $\dot{\varepsilon}_x$, $\dot{\varepsilon}_y$, $\dot{\gamma}$ and θ we shall seek expressions for the coefficients c_{ij}. The following reasoning is valid provided the eigenvalues are real and distinct.

Firstly we note that at $t = 0$ eq. (31) becomes

$$(35)\quad \begin{aligned} &(a)\quad c_{11} + c_{12} = x_o, \\ &(b)\quad c_{21} + c_{22} = y_o, \end{aligned}$$

where x_o and y_o are the initial coordinates to the particles we wish to follow. But we need two more independent equations to determine the four unknown coefficients. It is known (e.g. Kreider et al. 1968) that the ratio c_{11}/c_{21} equals the ratio between the components of the eigenvector that belongs to the eigenvalue $\varkappa_1$, and that the ratio c_{12}/c_{22} equals the ratio between the components of the eigenvector that belongs to the eigenvalue $\varkappa_2$. This is expressed in eqs. (36) and (37).

$$(36)\quad \begin{bmatrix} (a_{11} - \varkappa_1) & a_{12} \\ a_{21} & (a_{22} - \varkappa_1) \end{bmatrix} \begin{bmatrix} c_{11} \\ c_{21} \end{bmatrix} = 0,$$

$$(37)\quad \begin{bmatrix} (a_{11} - \varkappa_2) & a_{12} \\ a_{21} & (a_{22} - \varkappa_2) \end{bmatrix} \begin{bmatrix} c_{12} \\ c_{22} \end{bmatrix} = 0.$$

From eqs. (35), (36) and (37) we obtain

$$(38)\quad \begin{aligned} &(a)\quad c_{11} = \frac{(\varkappa_2 - a_{11})x_o - a_{12}y_o}{\varkappa_2 - \varkappa_1}, \\ &(b)\quad c_{12} = \frac{(a_{11} - \varkappa_1)x_o + a_{12}y_o}{\varkappa_2 - \varkappa_1}, \\ &(c)\quad c_{21} = \frac{(\varkappa_1 - a_{11})(\varkappa_2 - a_{11})a_{12}^{-1}x_o - (\varkappa_1 - a_{11})y^o}{\varkappa_2 - \varkappa_1}, \\ &(d)\quad c_{22} = \frac{(a_{11} - \varkappa_1)(\varkappa_2 - a_{11})a_{12}^{-1}x_o + (\varkappa_2 - a_{11})y_o}{\varkappa_2 - \varkappa_1}. \end{aligned}$$

When the expressions for a_{ij} and $\varkappa_i$ (formulas (32) and (34)) are inserted in the formulas for c_{ij} we see that the coordinates x and y as expressed by eq. (31) are uniquely determined by the initial coordinates x_o and y_o, by the strain rates $\dot{\varepsilon}$ and $\dot{\gamma}$, by the direction of simple shear relative to the x axis, θ, and of course by the lapse of time, t, after the commencement of the straining.

To gain insight into the character of the deformation a numerical example will be treated. In so doing we shall take care to ensure that the numerical value selected give eigenvalues that are both real and distinct because that is a necessary condition for the above mathematical treatment. Later in this paper the implication of complex eigenvalues shall be studied. Firstly one notes that the term $\frac{1}{2}(a_{11} + a_{22})$ in the eigenvalue formula (34) vanishes because $\dot{\varepsilon}_x = -\dot{\varepsilon}_y$ for incompressible materials undergoing plane strain. Only the root-sign term therefore remains in the eigenvalue whose expression reduces to

$$(39)\quad \varkappa_i = \pm\sqrt{\dot{\varepsilon}_x^2 - 2\dot{\varepsilon}_x\dot{\gamma}\sin\theta\cos\theta}$$

when the formulas for a_{11}, a_{12}, a_{21} and a_{22} are introduced and $\dot{\varepsilon}_y$ is put equal to $-\dot{\varepsilon}_x$. Real and distinct eigenvalues consequently occur either when

$$\dot{\varepsilon}_x > 2\dot{\gamma}\sin\theta\cos\theta$$

or when $\quad \dot{\varepsilon}_x < 0,$

provided that $\dot{\gamma}$ is positive (i.e. the displacement in the direction of positive x increases when y increases) and that the direction for the simple-shear component of the composite strain lies in the first and third quadrants. The latter condition implies that $\sin\theta\cos\theta$ is positive.

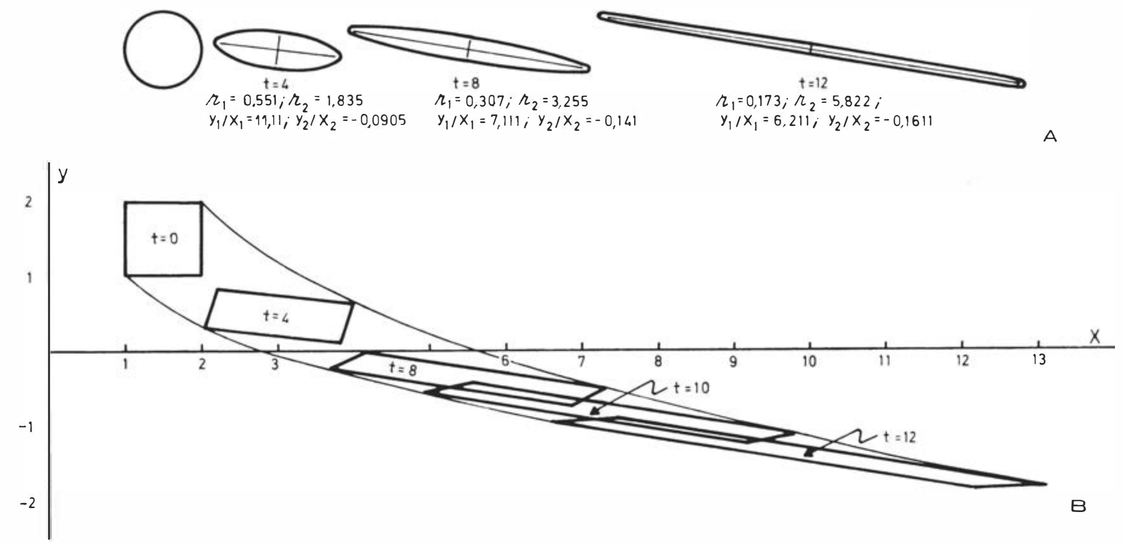


Fig. 3. Progressive deformation shown at different times of square deformed in a combination of pure shear and simple shear, see text. Particle paths of the corner points also shown, so is the strain ellipse. The origin of the coordinate system is fixed in the deforming body.

The conditions for real eigenvalues are met by the following selected quantities in a numerical example:

$$\theta = 45^\circ$$
$$\dot{\varepsilon}_x = -\dot{\varepsilon}_y = 0{,}2 \text{ time unit}^{-1}$$
$$\dot{\gamma} = 0{,}1 \text{ time unit}^{-1}.$$

The corresponding eigenvalues are

$$\varkappa_i = \pm\sqrt{0.02} = \pm 0{,}14142.$$

Inserted in the formula for c_{ij} listed above the selected quantities yield

$$c_{11} = 1{,}03033x_o + 0{,}17678y_o,$$
$$c_{12} = -0{,}03033x_o - 0{,}17678y_o,$$
$$c_{21} = -0{,}17678x_o - 0{,}03033y_o,$$
$$c_{22} = 0{,}17678x_o + 1{,}03033y_o.$$

Here x_o and y_o are the initial coordinates to the particle we wish to follow during the deformation.

The above expressions for $\varkappa_i$ and c_{ij} go into eq. (31) which then expresses quantitatively the particle path in terms of the coordinates x and y as functions of time, see eq. (40).

$$(40)\quad \begin{aligned} &\text{(a)}\ x = (1{,}03033x_o + 0{,}17678y_o)\exp(0{,}14142t) - \\ &\qquad - (0{,}03033x_o + 0{,}17678y_o)\exp(-0{,}14142t), \\ &\text{(b)}\ y = (-0{,}17678x_o - 0{,}03033y_o)\exp(0{,}14142t) + \\ &\qquad + (0{,}17678x_o + 1{,}03033y_o)\exp(-0{,}14142t). \end{aligned}$$

Let us follow firstly the movement and change of shape of an initial square whose edges were parallel to the coordinate axes and whose corners were initially located at the poins (1,1), (1,2), (2,1) and (2,2) respectively (Fig. 3).

In Fig. 3B are shown the particle paths of the corner points through the time $t = 0 \rightarrow t = 12$ time units. The distorted square is shown at time $t = 4$, 8, 10 and 12 units.

The strain ellipse, however, gives more immediate and detailed information on the character of strain than any other geometric figure. It is therefore worth taking the additional mathematical labor needed to determine the progressive changes of the strain ellipse.

An initial circle with unit radius is described by the expression

$$(41)\qquad x_o^2 + y_o^2 = 1$$

if its center coincides with the origin of the coordinate system.

To follow mathematically the deformation of the initial circle to an ellipse with continuously changing axial ratio we proceed as described above (p. 39 ff.) and solve the equations for the particle path (eq. (31)) with respect to x_o and y_o. The initial coordinates thus explicitly expressed in terms of the finite coordinates x and y at any time, t, are now emplaced in the equation for the initial circle which consequently changes to the equation for the strain ellipse at any time we choose to consider.

With the expressions for c_{ij} inserted, eq. (31) takes the form

$$(42)\quad \begin{aligned} &(a) \qquad x = Ax_o + Bx_o,\\ &(b) \qquad y = Dx_o + Ey_o, \end{aligned}$$

where we have used the notation:

$$(43)\quad \begin{aligned} A &= \frac{\varkappa_2 - a_{11}}{\varkappa_2 - \varkappa_1}\exp(\varkappa_1 t) - \frac{\varkappa_1 - a_{11}}{\varkappa_2 - \varkappa_1}\exp(\varkappa_2 t),\\ B &= \frac{-a_{12}}{\varkappa_2 - \varkappa_1}\big(\exp(\varkappa_1 t) - \exp(\varkappa_2 t)\big),\\ D &= \frac{(\varkappa_1 - a_{11})(\varkappa_2 - a_{11})}{(\varkappa_2 - \varkappa_1)a_{12}}\big(\exp(\varkappa_1 t) - \exp(\varkappa_2 t)\big),\\ E &= -\left(\frac{\varkappa_1 - a_{11}}{\varkappa_2 - \varkappa_1}\exp(\varkappa_1 t) - \frac{\varkappa_2 - a_{11}}{\varkappa_2 - \varkappa_1}\exp(\varkappa_2 t)\right) \end{aligned}$$

Solved for x_o and y_o the system (42) yields

$$(44)\quad \begin{aligned} &(a) \qquad \varkappa_o = \frac{Ex - By}{AE - BD},\\ &(b) \qquad y_o = \frac{Ay - Dx}{AE - BD}, \end{aligned}$$

which emplaced in the circle eq. (41) furnishes us with an expression for the strain ellipse:

$$(45)\quad \frac{E^2 + D^2}{(AE - BD)^2}x^2 - 2\frac{BE + AD}{(AE - BD)^2}xy + \frac{A^2 + B^2}{(AE - BD)^2}y^2 = 1.$$

This is the equation for an ellipse (or another conic depending upon the character of the coefficients) with center in origin and axes generally inclined to the coordinate axes. For a system with given $\dot{\varepsilon}$, $\dot{\gamma}$ and θ the eigenvalues $\varkappa_1$ and $\varkappa_2$ are also determined, and the coefficients A, B etc. depend only upon the time t. Hence eq. (45) with the expressions for the coefficients introduced shows in fact how the ellipse rotates and becomes deformed in the course of the progressive deformation. To show this explicitly we proceed to determine the functions between time and the orientation as well as the lengths of the ellipse axes.

Following the procedure previously applied the equation for the ellipse is now put in matrix form:

$$(46)\quad (x\ y)\begin{bmatrix} \dfrac{E^2 + D^2}{(AE - BD)^2} & -\dfrac{BE + AD}{(AE - BD)^2}\\ -\dfrac{BE + AD}{(AE - BD)^2} & \dfrac{A^2 + B^2}{(AE - BD)^2} \end{bmatrix}\begin{bmatrix} x\\ y \end{bmatrix} = 1.$$

In this form the equation for the ellipse furnishes both the length of the axes and their orientation by means of the above mentioned standard procedure of determining the eigenvalues and the eigenvectors. The eigenvectors that belong to the two eigenvalues of the coefficient matrix in eq. (46) coincide with the orientation of the axes of the ellipse, and the lengths of the axes are simply $r_1 = \frac{1}{\sqrt{\lambda_1}}$ and $r_2 = \frac{1}{\sqrt{\lambda_2}}$ (see p. 39). We choose λ as the symbol for the eigenvalues connected with determining the strain ellipse to avoid confusion with the eigenvalues $\varkappa$ used to determine the particle path; see p. 41. The eigenvalues of the coefficient matrix in eq. (46) are the roots of the quadratic characteristic equation, hence:

$$(47)\quad \lambda_i = \frac{1}{2(AE - BD)^2}\Big(A^2 + B^2 + D^2 + E^2 \pm \pm\sqrt{(A^2 + B^2 - D^2 - E^2)^2 + 4(BE + AD)^2}\Big).$$

The ratios between the components y and x of the eigenvectors — and therefore also the inclination of the axes of the ellipse — are

$$(48)\quad \begin{aligned} &(a) \qquad \mathrm{tg}\Phi_1 = \frac{y_1}{x_1} = \frac{E^2 + D^2 - \lambda_1}{AD + BE},\\ &(b) \qquad \mathrm{tg}\Phi_2 = \frac{y_2}{x_2} = \frac{E^2 + D^2 - \lambda_2}{AD + BE}. \end{aligned}$$

Φ are here the angles between the axes of the ellipse and the coordinate axis x.

Fig. 3A shows how an initial circle changes into a strain ellipse whose axial ratio and axial slope change continuously in the course of time. The values used for $\dot{\varepsilon}_x$, $\dot{\gamma}$ and θ are defined on p. 42.

Complex eigenvalues of the matrix of the rate-of-displacement equation: periodic particle path and "pulzating" strain ellipse. — For incompressible materials of the kind under study the eigenvalues to the matrix in the rate-of-displacement equation (30) contain only the square root part, thus

$$(49)\qquad \varkappa_i = \pm\tfrac{1}{2}\sqrt{\dot{\varepsilon}_x^2 - 2\dot{\varepsilon}_x\dot{\gamma}\sin\theta\cos\theta}.$$

If $\dot{\varepsilon}_x^2 < 2\dot{\varepsilon}_x\dot{\gamma}\sin\theta\cos\theta$, that is when $0 < \dot{\varepsilon}_x < 2\dot{\gamma}\sin\theta\cos\theta$, the eigenvalues are complex provided that *both* γ and the product $\sin\theta\cos\theta$ are either positive or negative. Remember that $\dot{\varepsilon}_x$ is always positive in our examples; see above. Since the general solution of the rate-of-displacement equation contains the eigenvalues as exponents, complex eigenvalues mean that the solution is of periodic form:

$$(50)\quad \begin{aligned}&(a)\quad x = c_{11}\cos(\beta t) + c_{12}\sin(\beta t), \text{ and}\\ &(b)\quad y = c_{21}\cos(\beta t) + c_{22}\sin(\beta t).\end{aligned}$$

Here $\beta = \sqrt{-\dot{\varepsilon}_x^2 + 2\dot{\varepsilon}_x\dot{\gamma}\sin\theta\cos\theta}$ and c_{ji} are coefficients.

That c_{11} and c_{21} equal x_o and y_o respectively, is readily seen when x and y are put equal to the initial coordinates x_o and y_o when $t = 0$. The two remaining coefficients are determined by differentiation of eq. (50) with respect to t and equating the differentiated forms to the initial expressions for $\dot{x}$ and $\dot{y}$ (eqs. (30) and (32)) at time zero. This procedure yields

$$(51)\quad \begin{aligned}&(a)\quad \dot{x} = \beta c_{12} = a_{11}x_o + a_{12}y_o,\\ &(b)\quad \dot{y} = \beta c_{22} = a_{21}x_o + a_{22}y_o.\end{aligned}$$

The consequent expressions for c_{12} and c_{22} are

$$c_{12} = \frac{a_{11}}{\beta}x_o + \frac{a_{12}}{\beta}y_o; \quad c_{22} = \frac{a_{21}}{\beta}x_o + \frac{a_{22}}{\beta}y_o.$$

The particle-path equations accordingly become

$$(52)\quad \begin{aligned}&(a)\quad x = \left[\cos(\beta t) + \frac{a_{11}}{\beta}\sin(\beta t)\right]x_o + \frac{a_{12}}{\beta}\sin(\beta t)y_o,\\ &(b)\quad y = \frac{a_{21}}{\beta}\sin(\beta t)x_o + \left[\cos(\beta t) + \frac{a_{22}}{\beta}\sin(\beta t)\right]y_o.\end{aligned}$$

These periodic equations describe particle paths that are closed in the sense that any particle will return to its starting point whenever (βt) is a multiple of 2π.

This solution looks intriguing and deserves further analysis. It obviously gives a deformation pattern quite unlike the straight particle paths occurring when the eigenvalues vanish (p. 47) or the curved but open paths implied by real eigenvalues (p. 42).

Before studying the behavior of the strain ellipse under the conditions of complex eigenvalues some particle paths will be calculated.

Let $\dot{\gamma} = 1$ and $\theta = 45^\circ$ as before. For $\dot{\varepsilon}_x$ we select 0,25 (which also implies $\dot{\varepsilon}_y = -0{,}25$) in order to make the eigenvalues complex. The selected parameters generate the following coefficients and eigenvalues

$$a_{11} = -0{,}25;\ a_{12} = 0{,}5;\ a_{21} = -0{,}5;\ a_{22} = 0{,}25,$$

$$\varkappa_i = \pm\sqrt{0{,}25^2 - 0{,}25} = \pm 0{,}433i = \pm\beta i.$$

These quantities are to be inserted into eq. (52) which consequently reads

$$(53)\quad \begin{aligned}&(a)\quad x = [\cos(0{,}433t) - 0{,}57737\sin(0{,}433t)]x_o + [1{,}1547\sin(0{,}433t)]y_o,\\ &(b)\quad y = -1{,}1547\sin(0{,}433t)x_o + [\cos(0{,}433t) + 0{,}57737\sin(0{,}433t)]y_o.\end{aligned}$$

The path traced by a particle originally at any given coordinate x_o and y_o is an ellipse whose axes bisect the coordinate axes such as shown in Fig. 4. Indeed, the paths of all particles in the body constitute a family of concentric ellipses, all with the same axial ratio and the same orientation, viz. the long axis making 45° angle with the positive x axis. The velocity along the paths relative to the fixed coordinate system increases proportional to the distance from origin as long as we are considering particles lying on the same radius. However, the velocity along any given path is not constant, neither is the path velocity

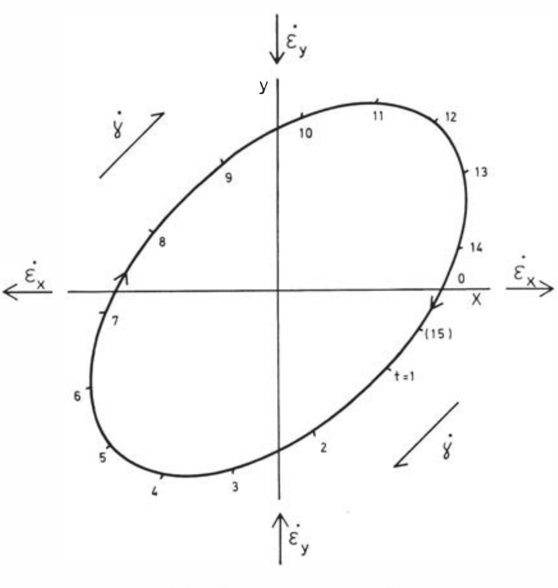


Fig. 4. Particle path of a particle initially at $x = 1$, $y = 0$. Positions shown at $t = 1$, 2, 3 etc. units of time. For the special combination of pure shear and simple shear needed for the closed path see text.

constant if we compare particles at the same distance from the center but at different radial angles. These relationships imply some strain in bodies affected by this particular combination of pure- and simple shear, but a chief part of the movements is a rigid rotation.

The interesting novelty in the present case which puts it in contrast to the cases with straight or curved but open particle paths is the complete round-the-clock rotation, the number of complete cycles only depending upon the time involved. In contrast the rotation in simple shear, for example, does not exceed 90° even if the magnitude of shear is infinite. Lines can never rotate across the simple shear direction.

It is informative to consider the movement pattern in two other numerical examples which give complex eigenvalues and thus closed particle paths.

In the first of these examples we select $\dot{\varepsilon}_x = 0{,}75$; $\dot{\gamma} = 1$ and $\theta = 45°$. The corresponding values for the coefficients etc. are

$$a_{11} = 0{,}25;\; a_{12} = 0{,}5;\; a_{21} = -0{,}5;\; a_{22} = -0{,}25.$$

$$\varkappa_i = \pm\sqrt{0{,}75^2 - 0{,}75} = \pm$$
$$\pm i\sqrt{0{,}1875} = \pm 0{,}433i = \pm\,\beta i.$$

The complex eigenvalues are identical to those obtained in the previous example, but since $\dot{\varepsilon}_x$ is different the equations controlling the particle path become slightly different, viz. (see also the general formula, eq. (52)):

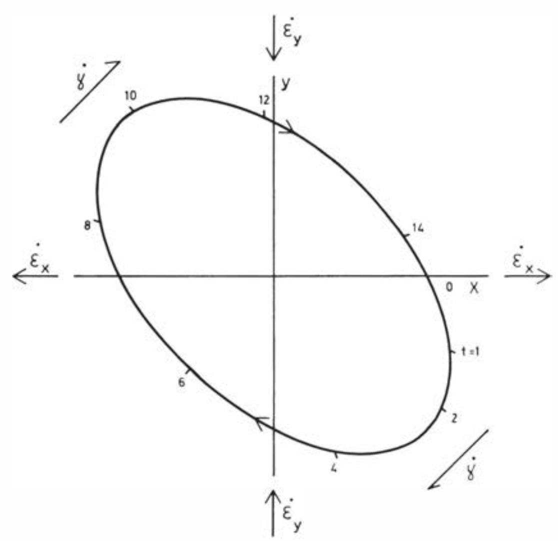


Fig. 5. Particle path analogous to the one shown in Fig. 4, but here the result of a different combination of pure shear and simple shear. See text.

$$(54)\quad \begin{aligned} &\text{(a)}\; x = [\cos(0{,}433t) + 0{,}57737\sin(0{,}433t)]x_o + \\ &\qquad + 1{,}1547\sin(0{,}433t)y_o, \\ &\text{(b)}\; y = -1{,}1547\sin(0{,}433t)x_o + \\ &\qquad + [\cos(0{,}433t) - 0{,}57737\sin(0{,}433t)]y_o. \end{aligned}$$

These equations describe particle paths belonging to a family of concentric and geometrically similar ellipses whose long axis makes an angle 135° with the positive x axis (Fig. 5). Though this set of particle path ellipses slope in the opposite direction of the ones above, the sense of rotation of the particles is the same, namely with the clock.

A special situation occurs at the following combination of simple shear and pure shear:

$$\dot{\varepsilon}_x = 0{,}5;\; \dot{\gamma} = 1;\; \theta = 45°.$$

From these data one obtains

$$a_{11} = 0;\; a_{12} = 0{,}5;\; a_{21} = -0{,}5;$$
$$a_{22} = 0 \text{ and } \varkappa_i = \pm 0{,}5i = \beta i.$$

Introduced into the particle-path equation these data yield

$$(55)\quad \begin{aligned} &\text{(a)}\quad x = \cos(0{,}5t)x_o + \sin(0{,}5t)y_o, \\ &\text{(b)}\quad y = -\sin(0{,}5t)x_o + \cos(0{,}5t)y_o. \end{aligned}$$

These are the equations for rigid rotation without any strain. That is, the particle paths constitute a family of concentric circles. The velocity on any given circle is constant but it increases proportional to the distance from the center.

We shall now turn to the behavior of the strain ellipse in the three cases above with complex eigenvalues of the matrix in the rate-of-deformation equation.

Putting the equations for the particle path (eqs. (52), (53), (54) and (55)) in the general form

$$(56)\quad \begin{aligned} &\text{(a)}\quad x = Ax_o + By_o, \\ &\text{(b)}\quad y = Dx_o + Ey_o. \end{aligned}$$

and going through the procedure of transforming the original circle to the strain ellipse expressed in matrix form, eq. (57), we obtain formula (58) for the eigenvalues.

$$(57)\quad (x\;\; y)\begin{pmatrix} \dfrac{D^2+E^2}{(AE-BD)^2} & -\dfrac{BE+AD}{(AE-BD)^2} \\ -\dfrac{BE+AD}{(AE-BD)^2} & \dfrac{A^2B^2}{(AE-BD)^2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 1,$$

$$(58)\quad \lambda_i = \tfrac{1}{2}\left[(A^2+B^2+D^2+E^2) \pm \right.$$
$$\left.\pm\sqrt{(A^2-E^2)^2+4(BE+AD)^2}\right].$$

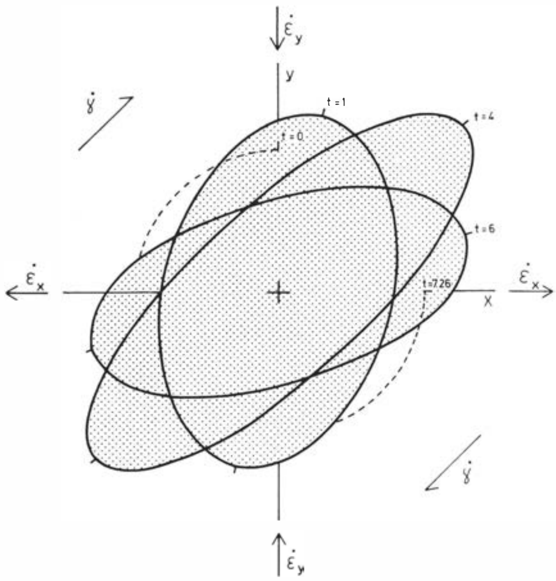


Fig. 6. Strain ellipse shown at $t = 1$, 4, 6 and 7,26 units of time when the combination of pure shear and simple shear is as in Fig. 4. Stipled curves indicate circumference of initial circle to which the strain ellipse will also return repeatedly at $t = n \times 7{,}26$ units of time. See text and Fig. 4.

Expression (58) is simpler than the one used previously (eq. (47)) because for the three cases now under discussion $AE - BD = 1$ and $B = -D$.

For the case $\dot{\varepsilon}_x = 0{,}25$, $\dot{\gamma} = 1$, $\theta = 45°$ we find

$$A = \cos(0{,}433t) - 0{,}57737 \sin(0{,}433t),$$
$$B = 1{,}1547 \sin(0{,}433t),$$
$$D = -1{,}1547 \sin(0{,}433t),$$
$$E = \cos(0{,}433t) + 0{,}57737 \sin(0{,}433t).$$

Inserted into expression (58) these coefficients give λ_1 and λ_2 as functions of time.

Based on the formulas for the length of the axes, viz.:

$$r_i = \frac{1}{\sqrt{\lambda_i}}$$

and for their slope

$$\mathrm{tg}\Phi_i = \frac{y_i}{x_i} = \frac{D^2 + E^2 - \lambda_i}{BE + AD}$$

we can obtain numerical values for the axial slope and the axial ratio at selected times during the progressive deformation.

Figure 6 refers to the combination $\dot{\varepsilon}_x = 0{,}25$, $\dot{\gamma} = 1$, $\theta = 45°$, and fig. 7 to the combination $\dot{\varepsilon}_x = 0{,}75$, $\dot{\gamma} = 1$, $\theta = 45°$. The only difference between the two cases is in the formulas for A and E. In the $\dot{\varepsilon}_x = 0{,}75$-case

$$A = \cos(0{,}433t) + 0{,}57737 \sin(0{,}433t),$$

and

$$\mathrm{E} = \cos(0{,}433t) - 0{,}57737 \sin(0{,}433t)$$

which differ from the values of A and E noted above. The plots in Figs. 6, 7, 8, and 9 give the length and orientation of the principal strain axes as functions of time.

For the combination $\dot{\varepsilon}_x = 0{,}5$, $\dot{\gamma} = 1$, $\theta = 45°$ the coefficients are

$$A = \cos(0{,}5t);\ B = \sin(0{,}5t);\ D = -\sin(0{,}5t);$$
$$E = \cos(0{,}5t).$$

Put into the matrix form of the ellipse, the latter coefficients give

$$(59) \qquad (x\ y)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 1.$$

The coefficient matrix of eq. (59) has two coincident eigenvalues equal to unity. Furthermore the slope of the axis, y/x, is $\frac{1-1}{0} = \frac{0}{0}$ which means that the axial orientation is indeterminate. We also note that the symbol for time has vanished for the strain ellipse equation which in fact has degenerated to the equation in matrix form of

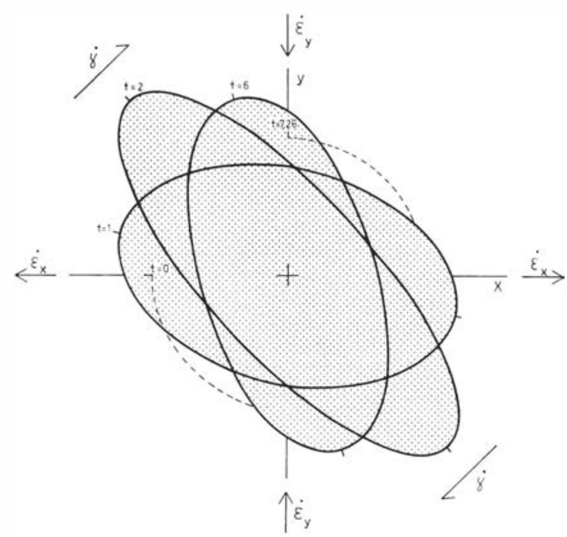


Fig. 7. Strain ellipse shown at $t = 1$, 2, 6 and 7,26 units of time when the combination of pure shear and simple shear is as in Fig. 5. Stipled curves indicate circumference of initial circle to which the strain ellipse returns periodically after $n \times 7{,}26$ units of time. See text and Fig. 5.

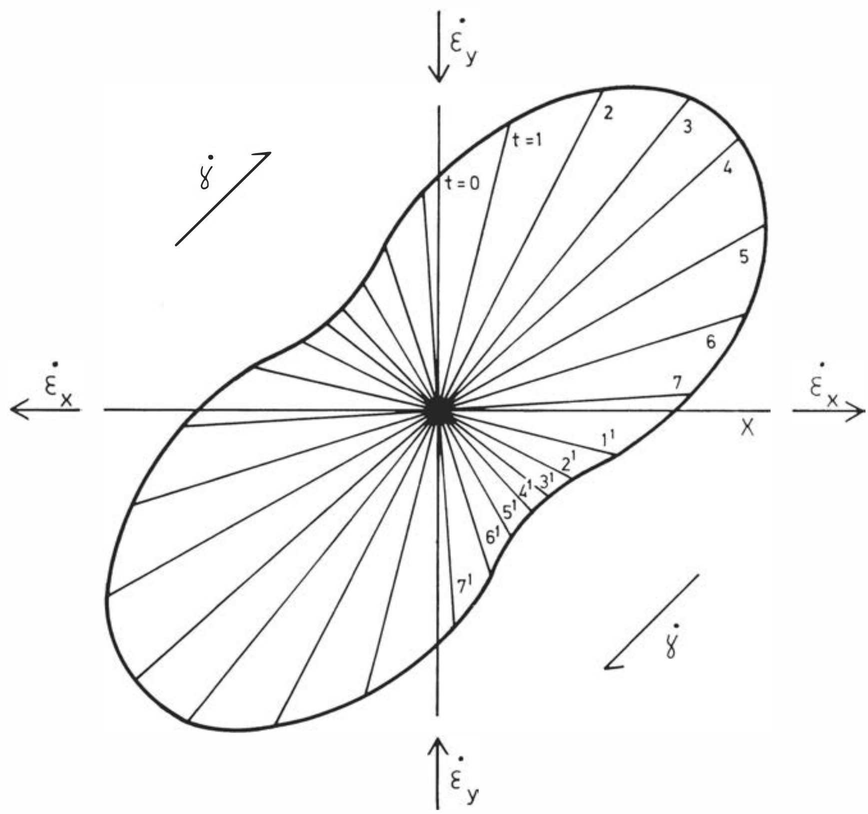


Fig. 8. Dumbbell-shaped pattern showing the orientation and length of the strain ellipse axes at $t = 0$, 1, 2, 3 etc. time units after the commencement of deformation. The radius of the initial circle equals the length cut off the y and x by the dumbbell-shaped curve. The radii marked $t = 0$, $t = 1$, 2 etc. and 1′, 2′ etc. mark the position and length of the principle axes of the strain ellipse at $t = 1$, 2, 3 etc. time units. The figure is to understand such that the radii with primed numbers coincide with the one principle axis and the unprimed radii coincide with the other axis. At $t = 3$ time units, for example, the long axis of the strain ellipse coincides with the radius marked 3 and the short axis falls on the radius marked 3′. Compare also Fig. 6.

a circle whose radius does not change with time. However, the linear transformation (55) which applies to the last example shows that the system undergoes a rigid rotation.

Vanishing eigenvalues of the matrix of the rate-of-displacement equation: straight-line particle paths. — The eigenvalues of the matrix of the rate-of-displacement equation (30) vanish when $\dot{\varepsilon}_x^2 = 2\dot{\varepsilon}_x\dot{\gamma}\sin\theta\cos\theta$ as noted on p. 41; that is when $\dot{\varepsilon}_x = 0$ and when $\dot{\varepsilon}_x = 2\dot{\gamma}\sin\theta\cos\theta$. The case $\dot{\varepsilon}_x = 0$ is trivial. It implies that the only deformation is the simple shear $\dot{\gamma}$ in the direction θ. The case $\dot{\varepsilon}_x = 2\dot{\gamma}\sin\theta\cos\theta$ is, however, worth some comments. For numerical demonstration we select $\theta = 45°$ and $\dot{\gamma} = 1$ as in the above examples. To satisfy the condition $\dot{\varepsilon}_x = 2\dot{\gamma}\sin\theta\cos\theta$ we must then give $\dot{\varepsilon}_x$ the value 1 (since $\sin 45° \cos 45° = 0{,}5$).

When the eigenvalues vanish for the above-mentioned matrix it can be shown that the solution to the rate-of-displacement equation (30) assumes the simple linear form (60)

$$(60)\quad \begin{aligned} &(a) \quad x = c_{11}t + c_{12}, \\ &(b) \quad y = c_{21}t + c_{22}. \end{aligned}$$

In these equations the constants $c_{12} = x_o$ and $c_{22} = y_o$. This is readily found by putting $t = 0$, and x and y equal to x_o and y_o respectively. As usual x_o and y_o are the initial coordinates. To

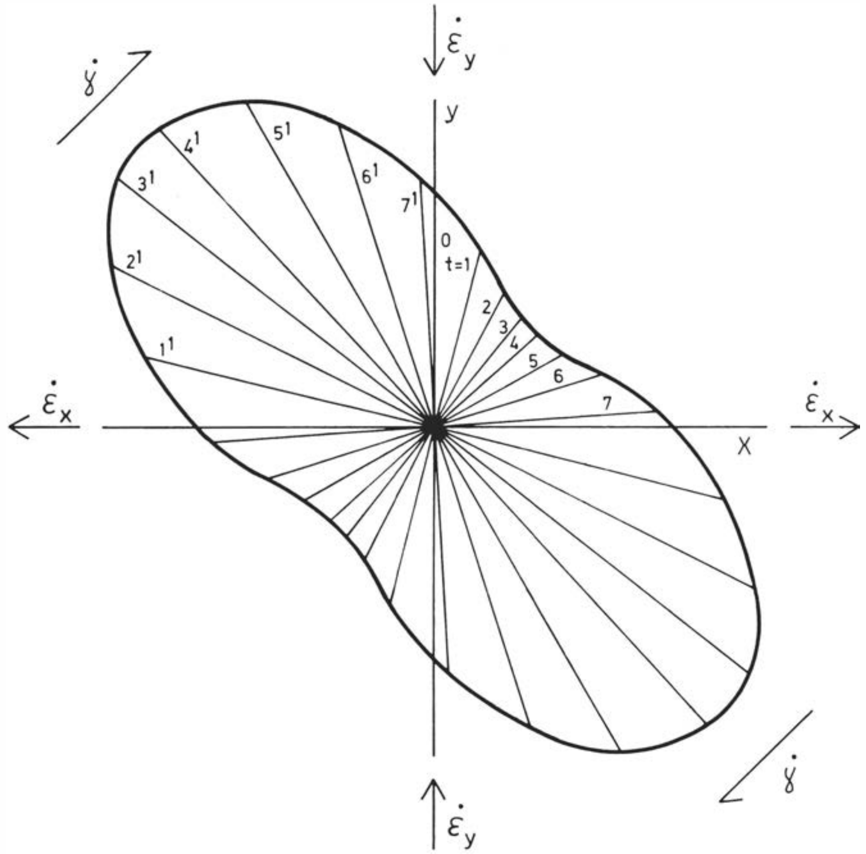


Fig. 9. Dumbbell-shaped pattern giving the length and orientation of the principal axes of the strain ellipse at $t=0$, 1, 2 etc. time units after commencement of deformation. The radius of the initial circle is equal to the length cut off the y and x axis by the dumbbell-shaped curve. For explanation see Fig. 8. Compare also Fig. 7.

determine c_{11} and c_{21} we differentiate with respect to t and obtain

$$\dot{x} = c_{11},$$
$$\dot{y} = c_{21}.$$

These expressions must also equal

$$\text{(a)} \qquad (\dot{\varepsilon}_x - \dot{\gamma}_o \sin\theta\cos\theta)x_o + (\dot{\gamma}\cos^2\theta)y_o$$

(61) and

$$\text{(b)} \qquad (-\dot{\gamma}\sin^2\theta)x_o + (\dot{\varepsilon}_y + \dot{\gamma}\sin\theta\cos\theta)y_o,$$

respectively, at time zero, see eq. (30).

Hence all four coefficients are determined, and eqs. (60) read

$$\text{(62)} \quad \begin{aligned} &\text{(a)} \quad x = [(\dot{\varepsilon}_x - \dot{\gamma}\sin\theta\cos\theta)x_o + (\dot{\gamma}\cos^2\theta)y_o]t + x_o, \\ &\text{(b)} \quad y = [(\dot{\varepsilon}_y + \dot{\gamma}\sin\theta\cos\theta)y_o - (\gamma\sin^2\theta)x_o]t + y_o. \end{aligned}$$

Introduction of the selected numerical quantities $\dot{\varepsilon}_x$, $\dot{\gamma}$, θ and rearrangement yield

$$\text{(63)} \quad \begin{aligned} &\text{(a)} \quad x = (1+\tfrac{1}{2}t)x_o + \tfrac{1}{2}ty_o, \\ &\text{(b)} \quad y = -\tfrac{1}{2}tx_o + (1-\tfrac{1}{2}t)y_o, \end{aligned}$$

in which we also have taken the condition for incompressibility, $\dot{\varepsilon}_x = -\dot{\varepsilon}_y$, into account. A study of the above equations reveals that they represent simple shear parallel to a direction that makes 135° with the x axis ($\theta' = 135°$) (Fig. 10). The shear strain is positive. In other words, the composite result of a simultaneous superposition of pure shear and simple shear of the magnitudes and relative orientation as selected in the present example is actually a new simple shear in a direction which is normal to the original direction

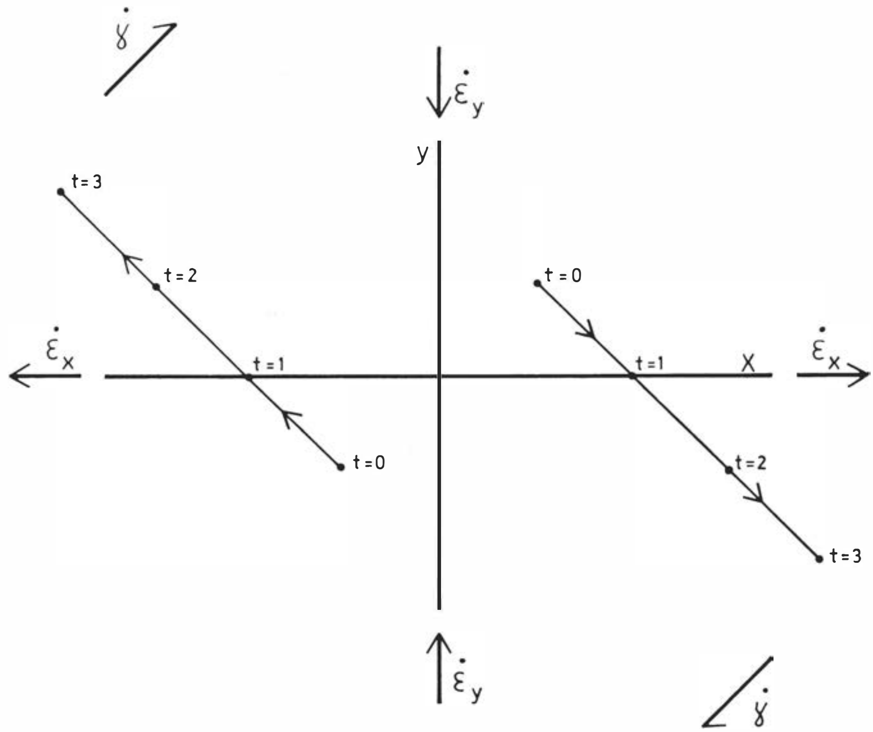


Fig. 10. Particle path (the lines from $t=0$ to $t=3$) of two points, initially at $t=0$, generated during combined pure shear and simple shear. For the relative magnitude and orientation of the two types of strain necessary to give the straight particle path see text.

and with a shear-strain rate equal to the original rate. In pure shear we know from general strain theory that the maximal shear-strain rate, $\dot{\gamma}_{max}$, in directions that bisects the x and y axes is numerically twice as large as the principal strain rates, $\dot{\varepsilon}_x$ and $\dot{\varepsilon}_y$. ($|\dot{\gamma}_{max}| = |2\dot{\varepsilon}_x|$). This is shown in Fig. 11A. When now a simple shear of magnitude $|\dot{\gamma}_\theta| = |\dot{\varepsilon}_x|$ is superimposed in a direction which makes 45° with the positive x axis the effect is that the shear-strain rates which are associated with the pure shear are reduced to half their original values. This is shown in Fig. 11B.

If we wish to obtain the equation for the strain ellipse corresponding to the deformation under study we follow the previously prescribed procedure. That is, we firstly solve eqs. (63) for the initial coordinates, thus

(64)
(a) $x_o = (1-\tfrac{1}{2}t)x - \tfrac{1}{2}ty,$

(b) $y_o = \tfrac{1}{2}tx + (1+\tfrac{1}{2}t)y,$

and insert these expressions for x_o and y_o into the equation for the initial circle with unit radius and center in origin. This procedure gives the strain ellipse equation

(65) $(\tfrac{1}{2}t^2 - t + 1)x^2 + t^2xy + (\tfrac{1}{2}t^2 + t + 1)y^2 = 1,$

whose matrix form is

(66) $$(x \;\; y)\begin{pmatrix} (\tfrac{1}{2}t^2 - t + 1) & \tfrac{1}{2}t^2 \\ \tfrac{1}{2}t^2 & (\tfrac{1}{2}t^2 + t + 1) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 1.$$

The eigenvalues of the coefficient matrix of eq. (66) are

(67) $$\lambda_i = \tfrac{1}{2}t^2 + 1 \pm \tfrac{1}{2}\sqrt{4t^2 + t^4}.$$

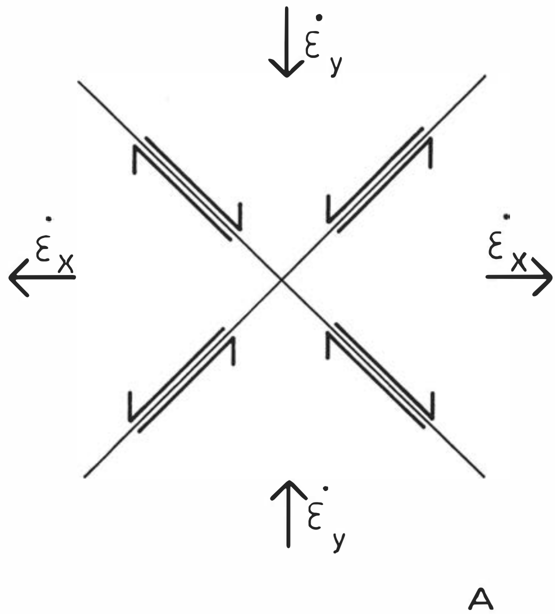
Applying the formulas

$$r_1 = \frac{1}{\sqrt{\lambda_1}}, \quad r_2 = \frac{1}{\sqrt{\lambda_2}},$$

(68)
(a) $$\mathrm{tg}\Phi_1 = \frac{y_1}{x_1} = \frac{\tfrac{1}{2}t^2 - t + 1 - \lambda_1}{-\tfrac{1}{2}t^2}$$

and

(b) $$\mathrm{tg}\Phi_2 = \frac{y_2}{x_2} = \frac{\tfrac{1}{2}t^2 - t + 1 - \lambda_2}{-\tfrac{1}{2}t^2},$$



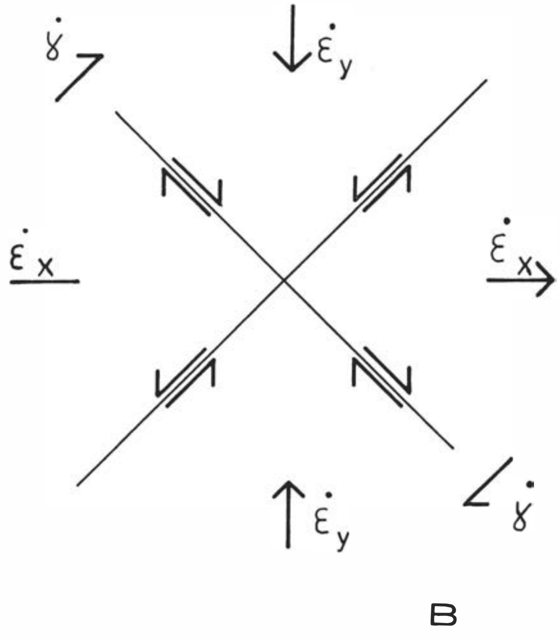


Fig. 11. Direction and sense for maximum instantaneous shear strain in the case of pure shear (*A*) and the case of combined pure shear and simple shear (*B*). See also text.

we find how the length and the slope of the axes of the strain change with time as the deformation proceeds.

The three-dimensional case

Sequential superposition of irrotational strain in three dimensions and simple shear

Irrotational finite strain in three dimensions of an incompressible substance can be described by the linear transformation (69) shown below provided that the axes of principal strain coincide with the coordinate axes.

$$(69)\qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} (1+\varepsilon_x) & 0 & 0 \\ 0 & (1+\varepsilon_y) & 0 \\ 0 & 0 & (1+\varepsilon_z) \end{bmatrix} \begin{bmatrix} x_o \\ y_o \\ z_o \end{bmatrix}.$$

Here x_o, y_o, z_o are the initial coordinates, x, y, z the final coordinates and ε_x, ε_y, ε_z the finite principal strains. The incompressibility of the material requires that $(1+\varepsilon_x)(1+\varepsilon_y)(1+\varepsilon_z)=1$.

A rotational part of the deformation can be introduced by adding simple shear to the three-dimensional irrotational strain. If we are free to determine the magnitude and the orientation of the simple shear relative to the x, y, z axes then the combination: three-dimensional irrotational strain and simple shear allows us to describe mathematically most — if not all — kinds of *homogeneous* three-dimensional strain of incompressible substances. The said combination is therefore of particular significance for the study of deformed rocks.

Simple shear strain of magnitude γ parallel to x', the plane x', y' being the slip plane, in an orthogonal coordinate system x', y', z' can be described by the linear transformation (70)

$$(70)\qquad \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & \gamma \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x'_o \\ y'_o \\ z'_o \end{bmatrix}.$$

Equation (70) describes a transformation in which there is no displacements parallel to y' and z' while the displacement in the x'direction increases with increasing distance from the x', y' plane. In order words, we have simple shear parallel to the x' axis. The motion may also be called laminar flow along x', the x', y' plane being parallel to the laminae which also coincide with the simple shear plane.

We have primed the coordinate system for the simple shear to distinguish it from the system used for the irrotational strain because the two coordinate systems need not be parallel when we combine the two kinds of deformation. It is now possible to add to the irrotational strain a simple shear of any magnitude and with any orientation of shear plane and shear direction simply by placing the axial system x', y', z' of eq. (70) in the appropriate orientation relative to the system x, y, z of eq. (69). In order to combine the two kinds of deformation we must transfer the coordinates of both transformations to a common coordinate system. We choose the x, y, z system of the irrotational strain as the common one; hence it only becomes necessary to transfer the coordinates in the x', y', z' system used for the simple shear into the x, y, z system.

The transformation of coordinates to points fixed in space between two orthogonal coordinate systems which are rotated in relation to one another about a common fixed origin, are given by eq. (71) which contains the matrix of direction cosines.

$$(71)\qquad \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos(x'x) & \cos(x'y) & \cos(x'z) \\ \cos(y'x) & \cos(y'y) & \cos(y'z) \\ \cos(z'x) & \cos(z'y) & \cos(z'z) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Here $(x'x)$, $(x'y)$ and $(x'z)$ are the angles between the x' axis and the axes x, y and z respectively. $(y'x)$, $(y'y)$ and $(y'z)$ are the angles between the y' axis and the axes x, y and z, while $(z'x)$, $(z'y)$ and $(z'z)$ are the angles between the axis z' and the x-, y- and z axes respectively. The square matrix in eq. (71) is the matrix of direction cosines also called the rotation matrix in three dimensions. If we wish to transform the simple shear as expressed in the x', y', z' system into the x, y, z system we must change both the initial coordinates and the final coordinates by means of the rotation matrix. Expressed in the x, y, z system the equation for the simple shear consequently becomes

$$(72)\qquad \begin{bmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{bmatrix} \begin{bmatrix} x_\gamma \\ y_\gamma \\ z_\gamma \end{bmatrix} = \begin{bmatrix} 1 & 0 & \gamma \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{bmatrix} \begin{bmatrix} x_{o\gamma} \\ y_{o\gamma} \\ z_{o\gamma} \end{bmatrix}.$$

In this expression $[\delta_{ij}]$ is the rotation matrix expressed somewhat more conveniently than in eq. (71). $x_{o\gamma}$, $y_{o\gamma}$, $z_{o\gamma}$, x_γ, y_γ and z_γ are the initial and the final coordinates respectively, in the x, y, z system of particles displaced by simple shear of magnitude γ in a direction x' which makes an angle θ_{11} (with cosine δ_{11}) with the x axis, an angle θ_{12} (with cosine δ_{12}) with the y axis and an angle θ_{13} (with cosine δ_{13}) with the z axis. The shear plane is normal to the z' axis, the latter making the angles θ_{31}, θ_{32} and θ_{33} with cosines δ_{31}, δ_{32} and δ_{33}, respectively, with the x avis, the y axis and the z axis.

We wish to express explicitly the final coordinates of a displaced particle in terms of the initial coordinates, the magnitude of finite simple shear and the orientation of the shear plane and shear direction, i.e. the direction cosines. To this end we firstly perform the matrix multiplication on the right-hand side of eq. (72). This procedure leads to eq. (73)

$$(73)\qquad \begin{bmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{bmatrix} \begin{bmatrix} x_\gamma \\ y_\gamma \\ z_\gamma \end{bmatrix} = \begin{bmatrix} (\delta_{11}+\delta_{31}\gamma) & (\delta_{12}+\delta_{32}\gamma) & (\delta_{13}+\delta_{33}\gamma) \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{bmatrix} \begin{bmatrix} x_{o\gamma} \\ y_{o\gamma} \\ z_{o\gamma} \end{bmatrix}.$$

In eq. (73) $x_{o\gamma}$, $y_{o\gamma}$ and $z_{o\gamma}$ may be regarded as the independent variables and x_γ, y_γ and z_γ as the dependent variables. The solution with respect to the latter are expressed in eq. (74)

(74)

$$\text{(a)}\qquad x_\gamma = \begin{vmatrix} a_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{vmatrix} x_{o\gamma} + \begin{vmatrix} a_{12} & \delta_{12} & \delta_{13} \\ \delta_{22} & \delta_{22} & \delta_{23} \\ \delta_{32} & \delta_{32} & \delta_{33} \end{vmatrix} y_{o\gamma} + \begin{vmatrix} a_{13} & \delta_{12} & \delta_{13} \\ \delta_{23} & \delta_{22} & \delta_{23} \\ \delta_{33} & \delta_{32} & \delta_{33} \end{vmatrix} z_{o\gamma},$$

$$\text{(b)}\qquad y_\gamma = \begin{vmatrix} \delta_{11} & a_{11} & \delta_{13} \\ \delta_{21} & \delta_{21} & \delta_{23} \\ \delta_{31} & \delta_{31} & \delta_{33} \end{vmatrix} x_{o\gamma} + \begin{vmatrix} \delta_{11} & a_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{vmatrix} y_{o\gamma} + \begin{vmatrix} \delta_{11} & a_{13} & \delta_{13} \\ \delta_{21} & \delta_{23} & \delta_{23} \\ \delta_{31} & \delta_{33} & \delta_{33} \end{vmatrix} z_{o\gamma},$$

$$\text{(c)}\qquad z_\gamma = \begin{vmatrix} \delta_{11} & \delta_{12} & a_{11} \\ \delta_{21} & \delta_{22} & \delta_{21} \\ \delta_{31} & \delta_{32} & \delta_{31} \end{vmatrix} x_{o\gamma} + \begin{vmatrix} \delta_{11} & \delta_{12} & a_{12} \\ \delta_{21} & \delta_{22} & \delta_{22} \\ \delta_{31} & \delta_{32} & \delta_{32} \end{vmatrix} y_{o\gamma} + \begin{vmatrix} \delta_{11} & \delta_{12} & a_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{vmatrix} z_{o\gamma}.$$

In eq. (74) $a_{11} \equiv \delta_{11}+\delta_{31}\gamma$; $a_{12} \equiv \delta_{12}+\delta_{32}\gamma$ and $a_{13} \equiv \delta_{13}+\delta_{33}\gamma$.

The straight vertical lines on either side of the array of direction cosines in eqs. (74) signify determinants. The solution for x_γ, y_γ and z_γ follows from the application of *Cramer's rule* (see e.g. Hadley 1965) on eq. (73) when the fact that the determinant to the matrix of direction cosines is unity is also considered (see e.g. Jaeger 1966).

For the benefit of the continued computation eqs. (74) are put in matrix form, thus:

$$\begin{pmatrix} x_\gamma \\ y_\gamma \\ z_\gamma \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} x_{o\gamma} \\ y_{o\gamma} \\ z_{o\gamma} \end{pmatrix}. \tag{75}$$

A_{11} etc. in eq. (75) correspond to the determinants $\begin{vmatrix} a_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{vmatrix}$ etc. in eqs (74).

It is also possible to develop an equation identical to (75) by making use of the condition that the inverse of the rotation matrix, $[\delta_{ij}]$, equals its transpose, $[\delta_{ij}]^*$. Equation (72) then takes the form:

$$\begin{pmatrix} x_\gamma \\ y_\gamma \\ z_\gamma \end{pmatrix} = [\delta_{ij}]^* \begin{pmatrix} 1 & 0 & \gamma \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} [\delta_{ij}] \begin{pmatrix} x_{o\gamma} \\ y_{o\gamma} \\ z_{o\gamma} \end{pmatrix}.$$

Here the product of the three square matrices on the right-hand-side of the equation is identical to the matrix $[A_{ij}]$ in eq. (75).

Equation (75) describes the displacement of particles in the x, y, z coordinate system when the shear direction and the shear plane of simple shear are inclined to the coordinate axes, the inclination been given by the magnitude of the elements in the matrix of direction cosines.

Now, in the same coordinate system an irrotational strain with principal strains parallel to the coordinate axes is described by eq. (76)

$$\begin{pmatrix} x_\varepsilon \\ y_\varepsilon \\ z_\varepsilon \end{pmatrix} = \begin{pmatrix} (1+\varepsilon_x) & 0 & 0 \\ 0 & (1+\varepsilon_y) & 0 \\ 0 & 0 & (1+\varepsilon_z) \end{pmatrix} \begin{pmatrix} x_{o\varepsilon} \\ y_{o\varepsilon} \\ z_{o\varepsilon} \end{pmatrix}. \tag{76}$$

Assume that the irrotational strain and the rotational simple shear are superimposed in different orders, viz. case (1): Irrotational strain is followed by simple shear, and case (2): Simple shear precedes irrotational strain.

Case (1): Irrotational strain precedes simple shear. — In this case the final coordinates to a particle after the irrotational deformation function as the initial coordinates to the same particle for the subsequent simple shear, hence we have

$$[x_{o\gamma}\, y_{o\gamma}\, z_{o\gamma}] = [x_\varepsilon\, y_\varepsilon\, z_\varepsilon]. \tag{77}$$

The consequent combination of eqs. (75) and (76) leads to eq. (78).

$$\begin{pmatrix} x_\gamma \\ y_\gamma \\ z_\gamma \end{pmatrix} = \tag{78}$$

$$= \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} (1+\varepsilon_x) & 0 & 0 \\ 0 & (1+\varepsilon_y) & 0 \\ 0 & 0 & (1+\varepsilon_z) \end{pmatrix} \begin{pmatrix} x_{o\varepsilon} \\ y_{o\varepsilon} \\ z_{o\varepsilon} \end{pmatrix}.$$

Matrix multiplication furnishes the total transformation from the initial coordinates of particles to their final coordinates after both the irrotational deformation and the simple shear have been performed. The result is presented in eq. (79).

$$\begin{pmatrix} x_\gamma \\ y_\gamma \\ z_\gamma \end{pmatrix} = \tag{79}$$

$$= \begin{pmatrix} A_{11}(1+\varepsilon_x) & A_{12}(1+\varepsilon_y) & A_{13}(1+\varepsilon_z) \\ A_{21}(1+\varepsilon_x) & A_{22}(1+\varepsilon_y) & A_{23}(1+\varepsilon_z) \\ A_{31}(1+\varepsilon_x) & A_{32}(1+\varepsilon_y) & A_{33}(1+\varepsilon_z) \end{pmatrix} \begin{pmatrix} x_{o\varepsilon} \\ y_{o\varepsilon} \\ z_{o\varepsilon} \end{pmatrix}.$$

Case (2): Simple shear precedes irrotational strain. — Now the final coordinates produced by the simple shear function as the initial coordinates to particles subsequently displaced in the irrotational strain. We therefore put

$$[x_{o\varepsilon}\, y_{o\varepsilon}\, z_{o\varepsilon}] = [x_\gamma\, y_\gamma\, z_\gamma] \tag{80}$$

and obtain the consequent combination of eqs. (75) and (76), viz.

$$\begin{pmatrix} x_\varepsilon \\ y_\varepsilon \\ z_\varepsilon \end{pmatrix} = \tag{81}$$

$$= \begin{pmatrix} (1+\varepsilon_x) & 0 & 0 \\ 0 & (1+\varepsilon_y) & 0 \\ 0 & 0 & (1+\varepsilon_z) \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} x_{o\gamma} \\ y_{o\gamma} \\ z_{o\gamma} \end{pmatrix}.$$

Carrying out the matrix multiplication we obtain eq. (82)

$$\begin{pmatrix} x_\varepsilon \\ y_\varepsilon \\ z_\varepsilon \end{pmatrix} = \tag{82}$$

$$= \begin{pmatrix} A_{11}(1+\varepsilon_x) & A_{12}(1+\varepsilon_x) & A_{13}(1+\varepsilon_x) \\ A_{21}(1+\varepsilon_y) & A_{22}(1+\varepsilon_y) & A_{23}(1+\varepsilon_y) \\ A_{31}(1+\varepsilon_z) & A_{32}(1+\varepsilon_z) & A_{33}(1+\varepsilon_z) \end{pmatrix} \begin{pmatrix} x_{o\gamma} \\ y_{o\gamma} \\ z_{o\gamma} \end{pmatrix}.$$

Since not all the three principal strains can be equal for incompressible media such as many rocks the equation systems (79) and (82) are not identical. The two equation systems accordingly show quantitatively how the composite deformation depends upon the order of superposition.

To obtain an equation for the finite strain ellipsoid we must solve eqs. (79) or (82) with respect to the initial coordinates x_o, y_o and z_o such that the latter become expressed in terms of the finite coordinates. Insertion of the thus formed expression for x_o, y_o and z_o into the equation for the initial sphere gives the equation for the finite strain ellipsoid.

Inversion of the matrix in eqs. (79) or (82) furnishes the sought expressions for x_o, y_o and z_o.

Let us note the elements in the inverted matrix by B and indicate the initial and final coordinates by the subscripts o and by no subscript respectively.

Then we have

$$(83)\qquad \begin{bmatrix} x_o \\ y_o \\ z_o \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

where $B_{ij} \equiv cof_{ji}\Delta^{-1}$, cof_{ji} being the cofactor to the element in the ji position in the matrix of eqs. (79) or (82) (depending on which order of superposition we have selected), and Δ the determinant of the same matrix. Incidentally, for incompressible substances the product $(1+\varepsilon_x)(1+\varepsilon_y)(1+\varepsilon_z)$ is unity such that the determinant of the coefficient matrices of both eqs. (79) and (82) reduce to

$$\begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix}$$

An initial sphere with unit radius and center in origin is described by eq. (84)

$$(84)\qquad x_o^2+y_o^2+z_o^2=1,$$

in which we introduce x_o etc. by their expressions given in eq. (83), and obtain:

(85)

$$(B_{11}^2+B_{21}^2+B_{31}^2)x^2+(B_{12}^2+B_{22}^2+B_{32}^2)y^2+$$
$$+(B_{13}^2+B_{23}^2+B_{33}^2)z^2+$$
$$+2(B_{11}B_{12}+B_{21}B_{22}+B_{31}B_{32})xy+$$
$$+2(B_{11}B_{13}+B_{21}B_{23}+B_{31}B_{33})xz+$$
$$+2(B_{12}B_{13}+B_{22}B_{23}+B_{32}B_{33})yz=1,$$

or more simply:

$$(86)\qquad ax^2+by^2+cz^2+2dxy+2exz+2fyz=1.$$

The latter two equations describe an ellipsoid with center at origin and axes inclined to the coordinate systems.

In order to determine the length and orientation of the axes of the ellipsoid we follow a procedure similar to the eigenvalue method used in the two-dimensional case, pp. 39 ff. Firstly the equation for the ellipsoid is put in matrix form:

$$(87)\qquad (x\ y\ z)\begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Here the elements a, b etc. are defined by the identy between eq. (85), eq. (86) and eq. (87).

Similarly as was done in the two-dimensional case the orientations of the ellipsoid axes in the three-dimensional system and the axial lengths can be determined by the eigenvalues of the coefficients matrix in eq. (87) see e.g. Efimov (1966). The length of the axes are

$$(88)\qquad r_1=\frac{1}{\sqrt{\lambda_1}};\ r_2=\frac{1}{\sqrt{\lambda_2}};\ r_3=\frac{1}{\sqrt{\lambda_3}};$$

where λ_i are the three eigenvalues of the 3×3 matrix in eq. (87). The eigenvectors which belong to the eigenvalues of the matrix coincide with the axes. These are conventional methods of analytical geometry and need not be verified in this account.

It is unfortunate that the equations which finally give us the shape and orientation of the strain ellipsoid viz. eqs. (85), (86) and (87) have quite cumbersome coefficients. These coefficients are the results of rather lengthy mathematical operations performed on the input data. The latter consist of ε_i, γ and the three independent angles θ_i needed to orient the simple shear direction and the simple shear plane in the coordinate system whose axes by choice coincide with the principal strains of the irrotational part of the composite deformation. The three angles mentioned are implied in the matrix of the nine direction cosines only three of which, however, are independent; see Jaeger 1966.

In view of the complexity of the coefficients of the strain-ellipsoid equation in the general case we shall select a special case with considerably simplified coefficients when we now present numerical examples.

In this special case the simple-shear plane coincides with the x, y plane but the shear direction is inclined to the x- and y axes (Fig. 12). This

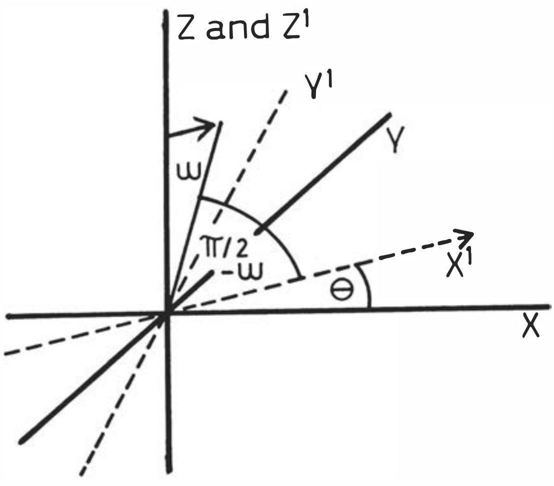


Fig. 12. Two orthogonal coordinate systems whose axes z and z' coincide and whose axes x and x', and y and y' respectively deviate by an angle θ.

orientation of the shear plane reduces the matrix of direction cosines to

$$(89)\qquad \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where θ is the angle between the shear direction and the x axis. That the element a_{33} is unity in this rotation matrix signifies that the z- and the z' axes coincide. A consequence of the present special combination of the two strains is that the matrix $[A_{ij}]$ in eqs. (75), (78) and (81) is quite simple. Inspection of eq. (74), in which the consequence of the rotation matrix is shown by expansion, reveals that the present version of the matrix $[A_{ji}]$ is

$$(90)\qquad \begin{bmatrix} 1 & 0 & \gamma\cos\theta \\ 0 & 1 & \gamma\sin\theta \\ 0 & 0 & 1 \end{bmatrix},$$

remembering that the determinants of the matrix of direction cosines is unity.

This in its turn reflects on the two transformations (79) and (82) valid for the composite deformation with the two contrasted orders of superposition. Transformation (79) is now of the form (91)

$$(91)\qquad \begin{bmatrix} x_\gamma \\ y_\gamma \\ z_\gamma \end{bmatrix} = \begin{bmatrix} (1+\varepsilon_x) & 0 & (1+\varepsilon_z)\gamma\cos\theta \\ 0 & (1+\varepsilon_y) & (1+\varepsilon_z)\gamma\sin\theta \\ 0 & 0 & (1+\varepsilon_z) \end{bmatrix} \begin{bmatrix} x_{o\varepsilon} \\ y_{o\varepsilon} \\ z_{o\varepsilon} \end{bmatrix},$$

which is valid when simple shear overprints irrotational strain. Transformation (82) takes the form (92) when the conditions for the special case are considered.

$$(92)\qquad \begin{bmatrix} x_\varepsilon \\ y_\varepsilon \\ z_\varepsilon \end{bmatrix} = \begin{bmatrix} (1+\varepsilon_x) & 0 & (1+\varepsilon_x)\gamma\cos\theta \\ 0 & (1+\varepsilon_y) & (1+\varepsilon_y)\gamma\sin\theta \\ 0 & 0 & (1+\varepsilon_z) \end{bmatrix} \begin{bmatrix} x_{o\gamma} \\ y_{o\gamma} \\ z_{o\gamma} \end{bmatrix}.$$

Equation (92) is accordingly valid when simple shear precedes irrotational strain.

As a numerical example we select the finite strains and the shear direction noted below.

$(1+\varepsilon_x) = 2$; $(1+\varepsilon_y) = 0{,}91$; $(1+\varepsilon_z) = 0{,}55$; $\gamma = 3{,}5$; $\theta = 60$.

Note that $(1+\varepsilon_x)(1+\varepsilon_y)(1+\varepsilon_z) = 1{,}001$ which means that the material is practically incompressible.

For the case that simple shear follows irrotational strain the selected parameters yield the transformation (93)

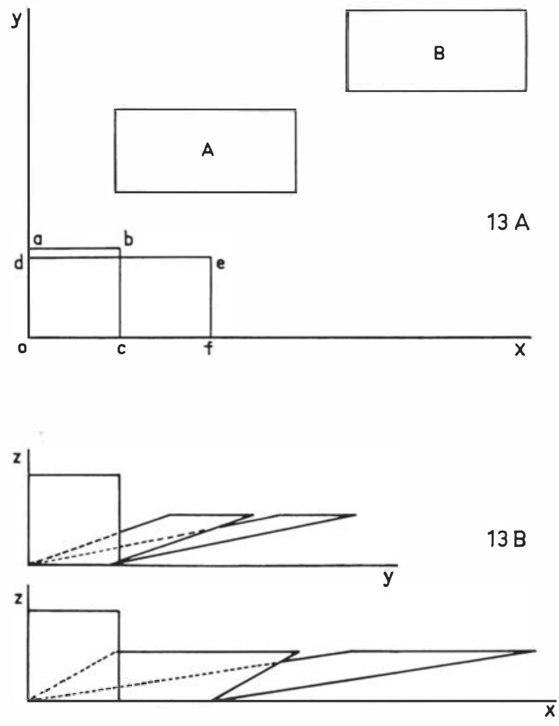


Fig. 13. Deformation of a cube in sequential superposition of 3-dimensional irrotational strain and simple shear of the kind explained in the text. 13A seen along the z axis toward the origin. *a b c o* are corners of the initial cube. *d e f o* are the corners of the base of the deformed cube. *A* is the top of the cube after deformation in the sequence irrotational strain → simple shear. *B* is the top of the deformed cube after deformation in the sequence, simple shear → irrotational strain. 13B shows profiles of the deformed cube in the planes z, y and z, x.

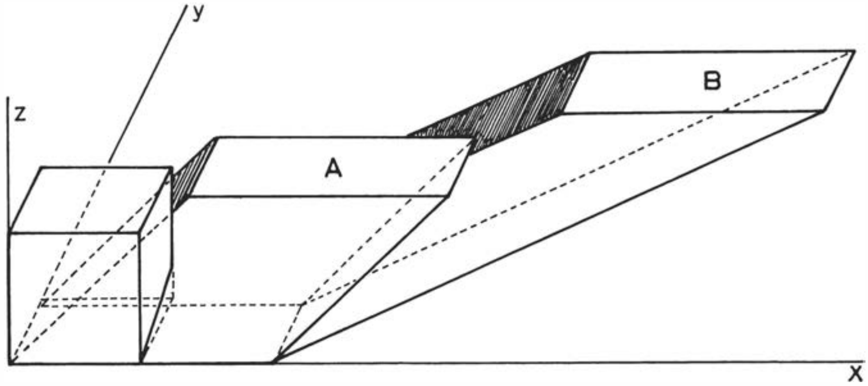


Fig. 14. The cube and its deformed versions as illustrated in Fig. 13 here shown in three dimensions.

$$(93)\quad \begin{bmatrix} x_\gamma \\ y_\gamma \\ z_\gamma \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0{,}9625 \\ 0 & 0{,}91 & 1{,}6671 \\ 0 & 0 & 0{,}55 \end{bmatrix} \begin{bmatrix} x_{o\varepsilon} \\ y_{o\varepsilon} \\ z_{o\varepsilon} \end{bmatrix}.$$

The deformation of a cube with corners initially at the points (000), (100), (010), (001), (101), (011), (110), (111) according to transformation (93) is demonstrated in Figs. 13 and 14.

If simple shear precedes irrotational strain the transformation assumes the form (94)

$$(94)\quad \begin{bmatrix} x_\varepsilon \\ y_\varepsilon \\ z_\varepsilon \end{bmatrix} = \begin{bmatrix} 2 & 0 & 3{,}5 \\ 0 & 0{,}91 & 2{,}758 \\ 0 & 0 & 0{,}55 \end{bmatrix} \begin{bmatrix} x_{o\gamma} \\ y_{o\gamma} \\ z_{o\gamma} \end{bmatrix}.$$

A cube of the initial shape and orientation described above will now change to the shape shown in Figs. 13 and 14.

For the determination of the strain ellipsoid in the special case with the simplified matrix of direction cosines we firstly need to invert the matrices of eqs. (91) and (92) in order to obtain explicit expressions for x_o, y_o and z_o. The results are given in eqs. (95) and (96).

Equation (95) is valid when irrotational strain precedes simple shear while eq. (96) holds for reverse sequence of superposition.

Introduction of these expressions for x_o, y_o and z_o into the equation for the initial sphere furnishes the sought equation for the strain ellipsoid, viz.:

$$(97)\quad ax^2+by^2+cz^2+2dxy+2exz+2fyz=1$$

(see eqs. (85) and (86).

The coefficients a, b, c etc. in eq. (97) are functions of the elements in the coefficient matrices either of eq. (95) or of eq. (96), depending upon the order of superposition. For the sequence: irrotational strain $\rightarrow$ simple shear the coefficients are as follows

$$\begin{aligned}
a_1 &= (1+\varepsilon_x)^{-2} = 0{,}2500, \\
b_1 &= (1+\varepsilon_y)^{-2} = 1{,}207584, \\
c_1 &= (1+\varepsilon_x)^{-2}\gamma^2\cos^2\theta + (1+\varepsilon_y)^{-2}\gamma^2\sin^2\theta = \\
&= 15{,}166085, \\
d_1 &= 0, \\
e_1 &= -(1+\varepsilon_x)^{-2}\gamma\cos\theta = -0{,}4375, \\
f_1 &= -(1+\varepsilon_y)^{-2}\gamma\sin\theta = -3{,}6602933.
\end{aligned}$$

The numerical values of the coefficients follow

$$(95)\quad \begin{bmatrix} x_{o\varepsilon} \\ y_{o\varepsilon} \\ z_{o\varepsilon} \end{bmatrix} = \begin{bmatrix} (1+\varepsilon_x)^{-1} & 0 & -(1+\varepsilon_x)^{-1}\gamma\cos\theta \\ 0 & (1+\varepsilon_y)^{-1} & -(1+\varepsilon_y)^{-1}\gamma\sin\theta \\ 0 & 0 & (1+\varepsilon_z)^{-1} \end{bmatrix} \begin{bmatrix} x_\gamma \\ y_\gamma \\ z_\gamma \end{bmatrix},$$

$$(96)\quad \begin{bmatrix} x_{o\gamma} \\ y_{o\gamma} \\ z_{o\gamma} \end{bmatrix} = \begin{bmatrix} (1+\varepsilon_x)^{-1} & 0 & -(1+\varepsilon_z)^{-1}\gamma\cos\theta \\ 0 & (1+\varepsilon_y)^{-1} & -(1+\varepsilon_z)^{-1}\gamma\sin\theta \\ 0 & 0 & (1+\varepsilon_z)^{-1} \end{bmatrix} \begin{bmatrix} x_\varepsilon \\ y_\varepsilon \\ z_\varepsilon \end{bmatrix}.$$

from the values selected for ε, γ and θ in our example specified on p. 42.

When the equation for the ellipsoid is put in matrix form (see eq. (87)) the characteristic equation which furnishes the eigenvalues is of the form

$$\lambda^3-(a+b+c)\lambda^2+(ab+ac+bc-d^2-e^2-f^2)\lambda+ \\ +af^2+be^2+cd^2-abc-2def=0. \tag{98}$$

With the numerical values of a_1, b_1, c_1 etc inserted, the characteristic equation becomes

$$\lambda^3-16{,}624\lambda^2+8{,}8186\lambda-0{,}9980=0, \tag{99}$$

whose three roots are (for the solution of cubic equations see for example Nagell 1962, p. 185)

$$\lambda_1=16{,}0630;\ \lambda_2=0{,}1624;\ \lambda_3=0{,}3822.$$

According to the relationship between axial length and eigenvalues (see p. 40) the lengths of the axes of the strain ellipsoid are:

$$r_1=\frac{1}{\sqrt{\lambda_1}}=0{,}24951;\ r_2=\frac{1}{\sqrt{\lambda_2}}=2{,}48145;$$

$$r_3=\frac{1}{\sqrt{\lambda_3}}=1{,}617565.$$

Since the initial sphere has unit radius and the materials are treated as incompressible the product $r_1r_2r_3$ should be unity. It is actually 1,0015.

The axes of the strain ellipsoid coincide with the eigenvectors that belong to the three eigenvalues. We accordingly extract the eigenvectors and find the following values for the relative components to the eigenvectors

$x_1=1$; $y_1=8{,}9057$; $z_1=-36{,}1439$ for eigenvector r_1,

$x_2=1$; $y_2=0{,}7012$; $z_2=0{,}2002$ for eigenvector r_2 and

$x_3=1$; $y_3=-1{,}3399$; $z_3=-0{,}3021$ for eigenvector r_3.

Only the relative values of the eigenvector components are given, based on the x component arbitrarily put equal to unity. The absolute lengths of the principal strain axes are determined above.

For comparison we consider also numerically the result of the opposite order of superposition of the same two kinds of strain. In this case some of the coefficients in the equation for the strain ellipsoid assume different forms and different magnitudes. As a consequence of the numerical values of the elements in the square matrix of eq. (96) the coefficients in the equation for the strain ellipsoid assume the values

$$a_2=(1+\varepsilon_x)^{-2}=0{,}2500,$$
$$b_2=(1+\varepsilon_y)^{-2}=1{,}2076,$$
$$c_2=(1+\varepsilon_z)^{-2}(\gamma^2\cos^2\theta+\gamma^2\sin^2\theta+1)= \\ =(1+\varepsilon_z)^{-2}(\gamma^2+1)=43{,}8017,$$
$$d_2=0,$$
$$e_2=-(1+\varepsilon_x)^{-1}(1+\varepsilon_z)^{-1}\gamma\cos\theta=-1{,}5909,$$
$$f_2=-(1+\varepsilon_y)^{-1}(1+\varepsilon_z)^{-1}\gamma\sin\theta=-6{,}0561.$$

Introduction of these numerical coefficients into the general equation for the strain ellipsoid (eq. 87) leads to the following characteristic equation

$$\lambda^3-45{,}2592\lambda^2+24{,}9389\lambda-0{,}9980=0. \tag{100}$$

The eigenvalues are accordingly found to be

$$\lambda_1=44{,}7018;\ \lambda_2=0{,}04344;\ \lambda_3=0{,}5140,$$

which determine the lengths of the axes, thus

$$r_1=\frac{1}{\sqrt{\lambda_1}}=0{,}1496;\quad r_2=\frac{1}{\sqrt{\lambda_2}}=4{,}7979;$$

$$r_3=\frac{1}{\sqrt{\lambda_3}}=1{,}3948.$$

(The product $r_1r_2r_3$ which theoretically should be unity is in fact 1,00114). The principal axes and the eigenvectors have the following relative components, putting arbitrarily the x component equal to unity:

$x_1=1$; $y_1=3{,}890$; $z_1=-27{,}941$ for the axis r_1,

$x_2=1$; $y_2=0{,}6755$; $z_2=0{,}1299$ for the axis r_2 and

$x_3=1$; $y_3=-1{,}449$; $z_3=-0{,}166$ for the axis r_3.

Simultaneous superposition of three-dimensional strain: progressive deformation in three dimensions

The simultaneous superposition of three-dimensional irrotational strain and a rotation caused by added simple shear can be treated similarly as the two-dimensional case, pp. 39 ff. For homogeneous irrotational deformation the rates of displacement $\dot{x}$, $\dot{y}$ and $\dot{z}$ are related to the rates of strain $\dot{\varepsilon}_x$, $\dot{\varepsilon}_y$, $\dot{\varepsilon}_z$ as follows

$$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{bmatrix}=\begin{bmatrix}\dot{\varepsilon}_x & 0 & 0\\ 0 & \dot{\varepsilon}_y & 0\\ 0 & 0 & \dot{\varepsilon}_z\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} \tag{101}$$

provided that the coordinate axes coincide with the principal axes of strain. As usual the dot above the symbols signifies differentiation with respect to time.

The rate of change of simple shear with displacement in the x' direction within the shear plane y', z' is related to the rates of displacement of the coordinates in three dimensions as shown below

$$(102)\qquad \begin{bmatrix} \dot{x}' \\ \dot{y}' \\ \dot{z}' \end{bmatrix} = \begin{bmatrix} 0 & 0 & \dot{\gamma} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}.$$

The axes x', y', z' are generally inclined to the axes x, y, z. If we wish to combine the two kinds of deformation we must therefore either rotate one of the two coordinate systems into the other or rotate both into a third common orientation without, of course, rotating the strains and displacements. We choose to rotate the system x', y', z' into x, y, z. This means that both the vector x', y', z' and the vector $\dot{x}', \dot{y}', \dot{z}'$ must be operated on by the rotation matrix in order to express the simple shear deformation in terms of coordinates in the system x, y, z. The angles in the rotation matrix then indicate the orientation of both the direction and the plane of simple shear in relation to the axes x, y, z, the latter coinciding with the axes of principal strain in the irrotational part of the composite deformation.

Application of the rotation matrix on $\dot{x}', \dot{y}', \dot{z}'$ and on x', y', z' in eq. (102) leads to

$$(103)\qquad \begin{bmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \dot{\gamma} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

where $[\delta_{ij}]$ is the rotation matrix (see eq. (72)). We now solve eq. (103) with respect to the rates of displacement $\dot{x}$ etc. and find

(104)

$$\dot{x} = \begin{vmatrix} \delta_{31}\dot{\gamma} & \delta_{12} & \delta_{13} \\ 0 & \delta_{22} & \delta_{23} \\ 0 & \delta_{32} & \delta_{33} \end{vmatrix} x + \begin{vmatrix} \delta_{32}\dot{\gamma} & \delta_{12} & \delta_{13} \\ 0 & \delta_{22} & \delta_{23} \\ 0 & \delta_{32} & \delta_{33} \end{vmatrix} y + \begin{vmatrix} \delta_{33}\dot{\gamma} & \delta_{12} & \delta_{13} \\ 0 & \delta_{22} & \delta_{23} \\ 0 & \delta_{32} & \delta_{33} \end{vmatrix} z,$$

$$\dot{y} = \begin{vmatrix} \delta_{11} & \delta_{31}\dot{\gamma} & \delta_{13} \\ \delta_{21} & 0 & \delta_{23} \\ \delta_{31} & 0 & \delta_{33} \end{vmatrix} x + \begin{vmatrix} \delta_{11} & \delta_{32}\dot{\gamma} & \delta_{13} \\ \delta_{21} & 0 & \delta_{23} \\ \delta_{31} & 0 & \delta_{33} \end{vmatrix} y + \begin{vmatrix} \delta_{11} & \delta_{33}\dot{\gamma} & \delta_{13} \\ \delta_{21} & 0 & \delta_{23} \\ \delta_{31} & 0 & \delta_{33} \end{vmatrix} z,$$

$$\dot{z} = \begin{vmatrix} \delta_{11} & \delta_{12} & \delta_{31}\dot{\gamma} \\ \delta_{21} & \delta_{22} & 0 \\ \delta_{31} & \delta_{32} & 0 \end{vmatrix} x + \begin{vmatrix} \delta_{11} & \delta_{12} & \delta_{32}\dot{\gamma} \\ \delta_{21} & \delta_{22} & 0 \\ \delta_{31} & \delta_{32} & 0 \end{vmatrix} y + \begin{vmatrix} \delta_{11} & \delta_{12} & \delta_{33}\dot{\gamma} \\ \delta_{21} & \delta_{22} & 0 \\ \delta_{31} & \delta_{32} & 0 \end{vmatrix} z.$$

The solutions for $\dot{x}$, $\dot{y}$ and $\dot{z}$ expressed in eqs. (104) follow from the application of *Cramer's rule* (Hadley 1965) on eq. (103) remembering that the determinant of the matrix of direction cosines $[\delta_{ij}]$ is always unity, see Jaeger (1966). Incidentally, we follow the convention of denoting determinants by a straight vertical line on either side of the array of elements.

A more convenient form of eqs. (104) is

$$(105)\qquad \begin{bmatrix} \dot{x}_\gamma \\ \dot{y}_\gamma \\ \dot{z}_\gamma \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} x_\gamma \\ y_\gamma \\ z_\gamma \end{bmatrix},$$

in which A_{ij} represent the determinants

$$\begin{vmatrix} \delta_{31}\dot{\gamma} & \delta_{12} & \delta_{13} \\ 0 & \delta_{22} & \delta_{23} \\ 0 & \delta_{32} & \delta_{33} \end{vmatrix} \text{ etc. in eqs. (104).}$$

Equation (105) can also be developed from (103) by using the transpose of the rotation matrix which for this particular matrix is equal to its inverse. Equation (103) can then be written:

$$\begin{bmatrix} \dot{x} \\ \dot{z} \\ \dot{y} \end{bmatrix} = [\delta_{ii}]^* \begin{bmatrix} 0 & 0 & \gamma \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} [\delta_{ij}] \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

The expression $[\delta_{ij}]^* \begin{bmatrix} 0 & 0 & \gamma \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} [\delta_{ii}]$

is identical to the matrix $[A_{ij}]$ in eq. (105).

If the rate of change of shear as well as the direction cosines are constant both in time and space we see that eqs. (104) or eq. (105) is in fact a system of linear first order differential equations with constant coefficients. The same is true with eq. (101) provided that the strain rates $\dot{\varepsilon}_x$, $\dot{\varepsilon}_y$ and $\dot{\varepsilon}_z$ are constant within the region considered. Now the combined effect of the irrotational strain and the simple shear is found by adding eq. (101) and eq. (105), thus

$$(106)\qquad \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} (A_{11}+\dot{\varepsilon}_x) & A_{12} & A_{13} \\ A_{21} & (A_{22}+\dot{\varepsilon}_y) & A_{23} \\ A_{31} & A_{32} & (A_{33}+\dot{\varepsilon}_z) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

A system of differential equations of this kind has the solution

$$(107)\qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} \exp(\varkappa_1 t) \\ \exp(\varkappa_2 t) \\ \exp(\varkappa_3 t) \end{bmatrix}.$$

Here $\varkappa_i$ are the eigenvalues of the coefficient matrix in the system of differential equations (106). If the three eigenvalues are distinct c_{ij} are constants while c_{ij} may be functions of t if the eigenvalues are coincident.

During the continued discussion we shall select special cases which simplify considerably the computation of the progressive deformation which for the completely general case requires quite cumbersome mathematical operations. As the first simple example let the z'- and z axes coincide and let θ be the angle between x and x'. This means that x, y is the plane of simple shear and that the shear direction makes an angle θ with the x axis (Fig. 12). In other words, the relative orientation of the two simultaneously superimposed strains is the same as in our example of sequential superposition. The here selected relative orientation corresponds to the rotation matrix (108)

$$(108)\qquad [\delta_{ij}] \equiv \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

When this rotation matrix is utilized to generate the elements A_{ij} in eq. (105) and (106) we find that eq. (106) assumes the form

$$(109)\qquad \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \dot{\varepsilon}_x & 0 & \dot{\gamma}\cos\theta \\ 0 & \dot{\varepsilon}_y & \dot{\gamma}\sin\theta \\ 0 & 0 & \dot{\varepsilon}_z \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

The eigenvalues of the coefficient matrix in eq. (109) are simply $\varkappa_1 = \dot{\varepsilon}_x$, $\varkappa_2 = \dot{\varepsilon}_y$ and $\varkappa_3 = \dot{\varepsilon}_z$ thus furnishing the integrated equation (110). Note that in the present relative orientation of the simple shear and the irrotational strain all three eigenvalues are real. A consequence of this is that the displacements are exponential functions of the time. (Cases with complex eigenvalues and hence periodic displacements will be treated later in this section.)

$$(110)\qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} \exp(\dot{\varepsilon}_x t) \\ \exp(\dot{\varepsilon}_y t) \\ \exp(\dot{\varepsilon}_z t) \end{bmatrix}.$$

The constants c_{ij} are partly determined by the initial coordinates x_o, y_o and z_o of a particle whose displacements we wish to trace, partly by the eigenvectors that belong to the three eigenvalues. It can be shown that the following sets of equations determine c_{ij}:

$$(111)\qquad \begin{aligned} c_{11}+c_{12}+c_{13} &= x_o, \\ c_{21}+c_{22}+c_{23} &= y_o, \\ c_{31}+c_{32}+c_{33} &= z_o, \end{aligned}$$

$$\text{(a)}\quad \begin{bmatrix} 0 & 0 & \dot{\gamma}\cos\theta \\ 0 & (\dot{\varepsilon}_y-\dot{\varepsilon}_x) & \dot{\gamma}\sin\theta \\ 0 & 0 & (\dot{\varepsilon}_z-\dot{\varepsilon}_x) \end{bmatrix} \begin{bmatrix} c_{11} \\ c_{21} \\ c_{31} \end{bmatrix} = 0,$$

$$(112)\ \text{(b)}\quad \begin{bmatrix} (\dot{\varepsilon}_x-\dot{\varepsilon}_y) & 0 & \dot{\gamma}\cos\theta \\ 0 & 0 & \dot{\gamma}\sin\theta \\ 0 & 0 & (\dot{\varepsilon}_z-\dot{\varepsilon}_y) \end{bmatrix} \begin{bmatrix} c_{12} \\ c_{22} \\ c_{32} \end{bmatrix} = 0,$$

$$\text{(c)}\quad \begin{bmatrix} (\dot{\varepsilon}_x-\dot{\varepsilon}_z) & 0 & \dot{\gamma}\cos\theta \\ 0 & (\dot{\varepsilon}_y-\dot{\varepsilon}_z) & \dot{\gamma}\sin\theta \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_{13} \\ c_{23} \\ c_{33} \end{bmatrix} = 0.$$

From these relationships we conclude

$$c_{12}=0,\ c_{21}=0,\ c_{31}=0,\ c_{32}=0,\ c_{33}=z_o,$$

$$c_{11}=x_o+\frac{\dot{\gamma}\cos\theta}{\dot{\varepsilon}_x-\dot{\varepsilon}_z}z_o;\quad c_{13}=\frac{\dot{\gamma}\cos\theta}{\dot{\varepsilon}_z-\dot{\varepsilon}_x}z_o,$$

$$c_{22}=y_o+\frac{\dot{\gamma}\sin\theta}{\dot{\varepsilon}_y-\dot{\varepsilon}_z}z_o;\quad c_{23}=\frac{\dot{\gamma}\sin\theta}{\dot{\varepsilon}_z-\dot{\varepsilon}_y}z_o.$$

Accordingly the integrated equations read:

(113)

$$\text{(a)}\quad x=\left(x_o+\frac{\dot{\gamma}\cos\theta}{\dot{\varepsilon}_x-\dot{\varepsilon}_z}z_o\right)\exp(\dot{\varepsilon}_x t)+\frac{\dot{\gamma}\cos\theta}{\dot{\varepsilon}_z-\dot{\varepsilon}_x}z_o\exp(\dot{\varepsilon}_z t),$$

$$\text{(b)}\quad y=\left(y_o+\frac{\dot{\gamma}\sin\theta}{\dot{\varepsilon}_y-\dot{\varepsilon}_z}z_o\right)\exp(\dot{\varepsilon}_y t)+\frac{\dot{\gamma}\sin\theta}{\dot{\varepsilon}_z-\dot{\varepsilon}_y}z_o\exp(\dot{\varepsilon}_z t),$$

$$\text{(c)}\quad z=z_o\exp(\dot{\varepsilon}_z t).$$

Let us select numerical values for $\dot{\varepsilon}$, $\dot{\gamma}$ and θ which make this case of simultaneous strain superposition comparable with a previous example on sequential superposition of irrotational three-dimensional strain and simple shear. For that example we used

$$(1+\varepsilon_x)=2;\ (1+\varepsilon_y)=0{,}91;\ (1+\varepsilon_z)=0{,}55;$$
$$\gamma=3{,}5 \text{ and } \theta=60^\circ$$

(see p. 54).

To make the rate of change of longitudinal strain in our present example comparable with the previous values for $(1+\varepsilon_x)$ etc. we shall recall the following relationship

$$(1+\varepsilon)=\frac{l}{l_o}=\exp(\dot{\varepsilon}t),$$

where l and l_o are the final and the initial lengths respectively. Hence the *ratio* between the strain rates which best compares with the above finite strains is

$$\dot{\varepsilon}_x/\dot{\varepsilon}_y/\dot{\varepsilon}_z=\ln(1+\varepsilon_x)/\ln(1+\varepsilon_y)/\ln(1+\varepsilon_z)= \\ =0{,}69315/-0{,}094311/-0{,}59784.$$

With reference to the shear strain, however, we use $\dot{\gamma}=3{,}5$ because the relationship between $\dot{\gamma}$ and γ is linear, viz.: $\gamma=\dot{\gamma}t$. θ is the same as in the earlier example, viz. 60°.

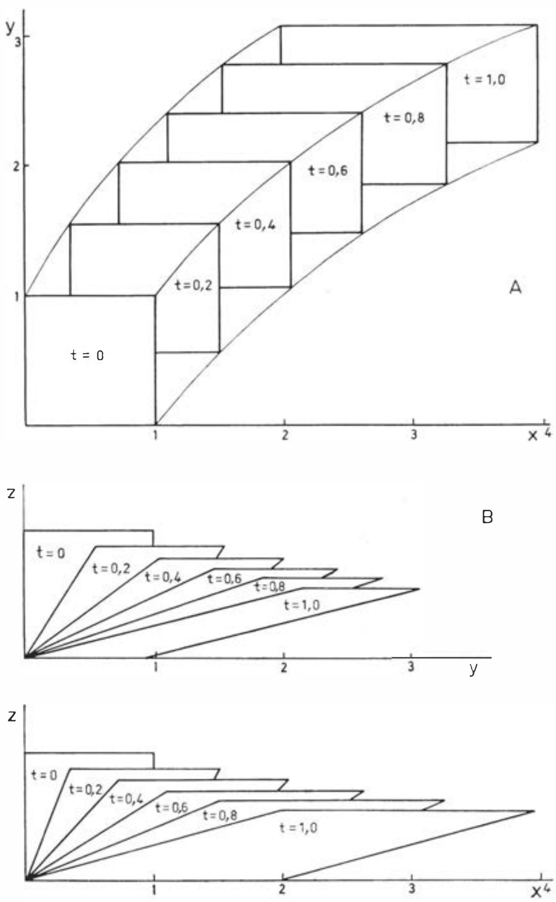


Fig. 15. Cube being deformed in simultaneous combination of 3-dimensional irrotational strain and simple shear. A: View along z axis toward origin. B: Profiles in the planes z, y and z, x.

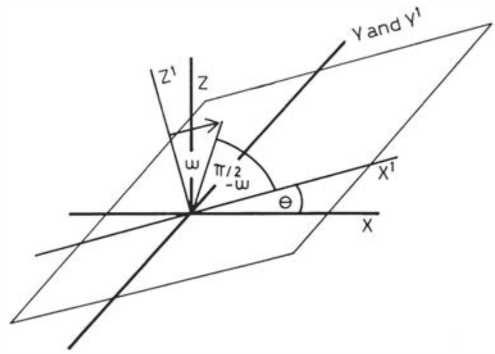


Fig. 16. Two orthogonal coordinate systems whose axes y and y' coincide and whose axes x and x' and z and z' respectively deviate by an angle θ. A plane containing the x' and y' or y axis is indicated.

With these values of $\dot{\varepsilon}$, $\dot{\gamma}$ and θ inserted eqs. (113) become

$$x=(x_o+1{,}356z_o)\exp(0{,}693t)- \\ -1{,}356z_o\exp(-0{,}598t),$$

$$y=(y_o+6{,}0177z_o)\exp(-0{,}0943t)- \\ -6{,}0177z_o\exp(-0{,}598t),$$

$$z=z_o\exp(-0{,}598t).$$

Fig. 15 A, B illustrates the progressive deformation of a cube with initial corners at (000), (100); (110); (010); (001); (101); (111); (011). It is interesting to compare the shape of the cube at time = 1 with the deformed shapes after the two sequential superpositions, Figs. 13 and 14. At time $t=1$ the finite strains $(1+\varepsilon_x)$, $(1+\varepsilon_y)$, $(1+\varepsilon_z)$ and γ are the same as in the sequential superposition yet, the shape produced by the simultaneous superposition is different from either of the two finite shapes produced by the two sequential superpositions of opposite order.

We shall proceed with another relative orientation of the simple shear and the irrotational strain. If the simple-shear plane is inclined to all three axial planes in the x, y, z system at the same time as the shear direction also deviates from the axial planes then we have the completely general case. This means that none of the elements in the rotation matrix vanishes (though many are identical since only three direction cosines in the three-dimensional rotation matrix are independent) and the computation becomes more cumbersome than the added information probably warrants. However, if we maintain a shear plane that is inclined to the x, y plane but let the shear direction lie in, say, the z, x plane (Fig. 16) we will get some interesting composite strain geometries without making the calculations excessively complicated. The orientation specified means that the y'- and y axes coincide and that the axes x and x' and z and z' respectively, deviate from one another by an angle θ. The primed coordinate axes refer to the simple shear in the manner defined by eq. (102). The corresponding rotation matrix to apply in order to express the coordinates x', y', z' in terms of the coordinates x, y, z is now

$$(114) \qquad [\delta_{ij}] \equiv \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}.$$

Using this rotation matrix in eq. (103) we obtain expressions for the elements A_{ij} in eqs. (105) and (106). The latter then is of the form:

$$(115) \qquad \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} =$$

$$= \begin{bmatrix} (\dot{\varepsilon}_x - \dot{\gamma}\sin\theta\cos\theta) & 0 & \dot{\gamma}\cos^2\theta \\ 0 & \dot{\varepsilon}_y & 0 \\ -\dot{\gamma}\sin^2\theta & 0 & (\dot{\varepsilon}_z + \dot{\gamma}\sin\theta\cos\theta) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

The coefficient matrix in eq. (115) has the characteristic equation

$$(116) \qquad \varkappa^3 + A\varkappa^2 + B\varkappa + C = 0.$$

Here $A = \dot{\varepsilon}_x + \dot{\varepsilon}_y + \dot{\varepsilon}_z = 0$ for incompressible materials to which we limit the treatment in this paper,

$$B = \dot{\varepsilon}_x\dot{\varepsilon}_y + \dot{\varepsilon}_x\dot{\varepsilon}_z + \dot{\varepsilon}_y\dot{\varepsilon}_z + (\dot{\varepsilon}_x - \dot{\varepsilon}_z)\dot{\gamma}\sin\theta\cos\theta,$$

and

$$C = (\dot{\varepsilon}_z - \dot{\varepsilon}_x)\dot{\varepsilon}_y\dot{\gamma}\sin\theta\cos\theta - \dot{\varepsilon}_x\dot{\varepsilon}_y\dot{\varepsilon}_z.$$

A cubic equation has generally three roots which may be real or complex. The previous combination of irrotational strain and simple shear with the shear plane coinciding with the x, y plane produced only characteristic equations with real roots (i.e. real eigenvalues). The present combination of the two classes of strain with the simple shear plane at an angle to the x, y plane has, however, characteristic equations whose roots may be complex. That is to say the integrated form (eq. (107)) of the rate-of-displacement equation is sometimes periodic in the sense that the particle paths extend around the full 360° angle.

We shall now consider numerical combinations of $\dot{\varepsilon}_x$, $\dot{\varepsilon}_y$, $\dot{\varepsilon}_z$, $\dot{\gamma}$ and θ which give complex eigenvalues.

To decide whether the roots of a cubic equation is real or complex one studies the quantity

$$D \equiv \left(\frac{C}{2}\right)^2 + \left(\frac{B}{3}\right)^3$$

(see e.g. Nagell 1962).

Here B and C are the coefficients in eq. (116) in which the coefficient A vanishes. The latter circumstance is necessary for the following conditions to be valid:

(1) When $D < 0$ all three roots are real and distinct.

(2) When $D = 0$ two roots are coincident and all roots are real.

(3) When $D > 0$ there is one real and two conjugate complex roots.

We shall focus our attention on case (3) which implies periodic displacement of particles in the course of progressive deformation. Before, however, presenting combinations of numerical values of $\dot{\varepsilon}_x$, $\dot{\varepsilon}_y$, $\dot{\varepsilon}_z$, $\dot{\gamma}$ and θ that give complex eigenvalues we note that the integrated rate-of-displacement equation (107) under such circumstances takes the following general form:

$$\text{(a)} \quad x = c_{x1}\exp(\varkappa_1 t) + \left(c_{x2}\cos(\beta t) + c_{x3}\sin(\beta t)\right)\exp(\alpha t),$$

$$(117) \text{ (b)} \quad y = c_{y1}\exp(\varkappa_1 t) + \left(c_{y2}\cos(\beta t) + c_{y3}\sin(\beta t)\right)\exp(\alpha t),$$

$$\text{(c)} \quad z = c_{z1}\exp(\varkappa_1 t) + \left(c_{z2}\cos(\beta t) + c_{z3}\sin(\beta t)\right)\exp(\alpha t).$$

Here $\varkappa_1$ is the real eigenvalue, α the real part of the two complex eigenvalues and $\pm\beta$ their imaginary parts. c_{x1}, c_{x2} etc. are coefficients partly controlled by the initial coordinates x_o, y_o and z_o, partly by the eigenvectors belonging to the three eigenvalues.

Now, at this point we shall make use of the condition that the 3×3 matrix in eq. (115) is of a special nature which permits us to extract the eigenvalues without going through the lengthy procedure of solving a third degree equation such as would be necessary in the general case. We shall presently see that the root $\varkappa_1 = \dot{\varepsilon}_y$ is found by inspection and that the two other roots, which may or may not be complex, are determined by a second degree equation. Denoting the elements in the matrix of eq. (115) by a_{ij} the characteristic equation may be written in the simple form

$$(118) \qquad (a_{11} - \varkappa)(a_{22} - \varkappa)(a_{33} - \varkappa) - a_{13}a_{31}(a_{22} - \varkappa) = 0,$$

because the four elements a_{12}, a_{21}, a_{23} and a_{32} in the matrix of eq. (115) are all zero. One of the roots which satisfies this equation is clearly

$$\varkappa_1 = a_{22} \quad (= \dot{\varepsilon}_y \text{ in our special case}).$$

Since $\dot{\varepsilon}_y$ is always real so is the root $\varkappa_1$.

If we moreover cancel $(a_{22} - \varkappa)$ on either side of the equality sign we obtain the quadratic equation

$$(119) \qquad (a_{11} - \varkappa)(a_{33} - \varkappa) - a_{13}a_{31} = 0,$$

whose two roots are

$$(120) \qquad \varkappa_i = \tfrac{1}{2}(a_{11} + a_{33}) \pm \pm \sqrt{\tfrac{1}{4}(a_{11} + a_{33})^2 + a_{13}a_{31} - a_{11}a_{33}}.$$

With the expressions for a_{ij} inserted the formula for the roots is

$$(121) \qquad \varkappa_i = \tfrac{1}{2}(\dot{\varepsilon}_x + \dot{\varepsilon}_z) \pm \pm \tfrac{1}{2}\sqrt{(\dot{\varepsilon}_x - \dot{\varepsilon}_z)^2 - 4(\dot{\varepsilon}_x - \dot{\varepsilon}_z)\dot{\gamma}\sin\theta\cos\theta}.$$

These two roots or eigenvalues are complex when the terms within the square-root sign satisfy the condition

$$(122) \qquad 0 < (\dot{\varepsilon}_x - \dot{\varepsilon}_z) < 4\dot{\gamma}\sin\theta\cos\theta,$$

(we assume that $\gamma\sin\theta\cos\theta$ is positive) in which case the imaginary part of the roots is

$$(123) \qquad i\beta = i\tfrac{1}{2}\sqrt{4(\dot{\varepsilon}_x - \dot{\varepsilon}_z)\dot{\gamma}\sin\theta\cos\theta - (\dot{\varepsilon}_x - \dot{\varepsilon}_z)^2},$$

and the real part

$$(124) \qquad \alpha = \tfrac{1}{2}(\dot{\varepsilon}_x + \dot{\varepsilon}_z).$$

Numerical example (1). — First we select the following strain rates and angle between the shear-plane and the x, y plane:

$$\dot{\varepsilon}_x = 0{,}25, \ \dot{\varepsilon}_y = 0{,}05, \ \dot{\varepsilon}_z = -0{,}30, \quad \dot{\gamma} = 1{,}0, \ \theta = 45°.$$

These values correspond to a weak lengthening in the y direction (i.e. along the axis of rotation) and an average shrinkage in the x, z plane.

Inserted in the formula for the eigenvalues these data give

$$\varkappa_1 = 0{,}05, \ \varkappa_2 = -0{,}025 + 0{,}446514i, \quad \varkappa_3 = -0{,}025 - 0{,}446514i.$$

Before introducing the eigenvalues into the integrated eqs. (117) we note that a number of the coefficients in these equations vanishes. Without going through the argument here we can show that c_{x1}, c_{z1}, c_{y2} and c_{y3} all vanish, hence the integration yields

$$(125) \qquad \begin{aligned} x &= e^{-0{,}025t}\big(c_{x2}\cos(0{,}446514t) + c_{x3}\sin(0{,}446514t)\big), \\ y &= c_{y1}e^{0{,}05t}, \\ z &= e^{-0{,}025t}\big(c_{z2}\cos(0{,}446514t) + c_{z3}\sin(0{,}446514t)\big). \end{aligned}$$

The five coefficients may be determined by the five independent equations below which are valid at $t = 0$

$$(126) \qquad \begin{aligned} &\text{(a)} \quad x_o = c_{x2}, \\ &\text{(b)} \quad y_o = c_{y1}, \\ &\text{(c)} \quad z_o = c_{z2}, \\ &\text{(d)} \quad \dot{x} = a_{11}x_o + a_{13}z_o = -0{,}25x_o + 0{,}5z_o = \\ &\qquad = -0{,}025c_{x2} + 0{,}446514c_{x3}, \\ &\text{(e)} \quad \dot{z} = a_{31}x_o + a_{33}z_o = -0{,}5x_o + 0{,}2z_o = \\ &\qquad = -0{,}025c_{z2} + 0{,}446514c_{z3}. \end{aligned}$$

The first three equations follow directly from eqs. (125) by simply putting $t = 0$ while the two last ones are obtained by differentiation with respect to t of eqs. (125) and equating the result with the original differential equation (115) in which the relevant numerical parameters are inserted. The coefficients c_{x2}, c_{x3}, c_{z2}, c_{z3} and c_{y1} thus determined are inserted in eqs. (125) to give their numerical forms

$$(127) \qquad \begin{aligned} x &= \big(x_o\cos(0{,}446514t) + (1{,}119786z_o - \\ &\quad - 0{,}5039x_o)\sin(0{,}446514t)\big)\,e^{(-0{,}025t)}, \\ y &= y_o e^{0{,}05t}, \\ z &= \big(z_o\cos(0{,}446514t) + (0{,}5039z_o - \\ &\quad - 1{,}119786x_o)\sin(0{,}446514t)\big)\,e^{(-0{,}025t)}. \end{aligned}$$

The particle paths as given by these equations are spirals whose diameter decreases continuously at the same time as they stretch out along the y- and $-y$ axes. Only points which originally were on the x, z plane remain on that plane while spiraling about origin. In Fig. 17 the general character of the particle paths is indicated. The period is $t = \dfrac{2\pi}{0{,}446514} = 14{,}072$ units of time. In the course of a period the particles get closer to the y axis — which is the axis of rotation — by an amount determined by the factor $e^{-0{,}025 \cdot 14{,}072} = e^{-0{,}3518} = 0{,}70342$. For example a particle originally at $x_o = 1$, $y_o = 0$, $z_o = 0$ is after one period at $x_o = 0{,}70342$, $y_o = 0$, $z_o = 0$.

Numerical example (2). — In this example we let the strain $\dot{\varepsilon}_y$ be compressive, accordingly there is an average expansion in the x, z plane in order to keep the volume constant. We select the follow-

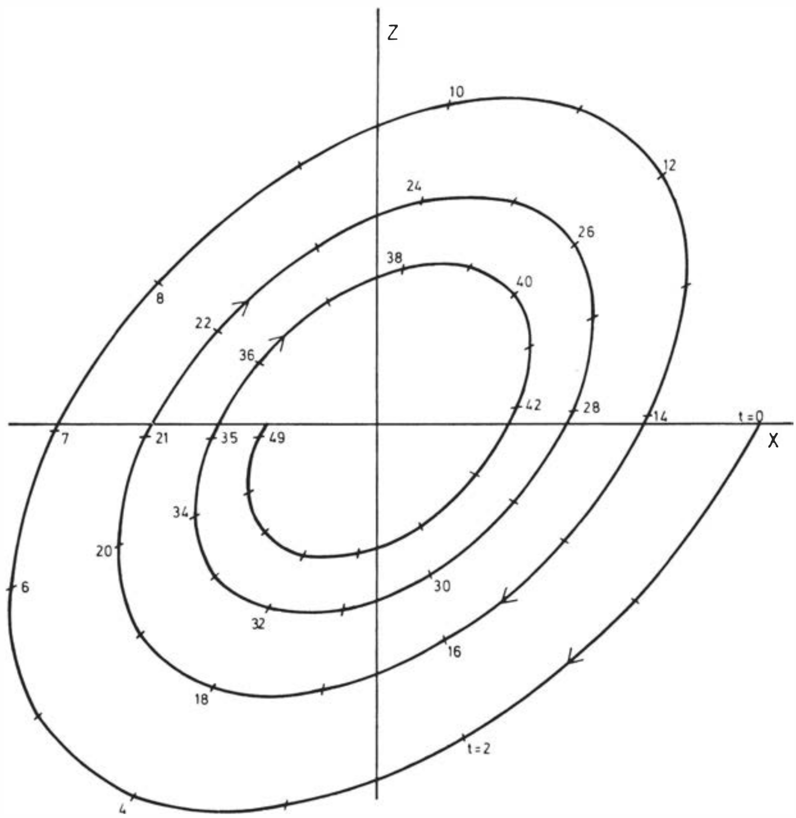


Fig. 17. Spiraling particle path in the plane $y=0$ characteristic for a certain combination of 3-dimensional strains. Described by a particle initially at $x=1$, $z=0$. Positions of particle indicated at $t=1$, 2, 3 etc. time units. For description see text.

ing values for the strain rates and the shear-plane inclination

$$\dot{\varepsilon}_x = 0{,}30,\ \dot{\varepsilon}_y = -0{,}05,\ \dot{\varepsilon}_z = -0{,}25,$$
$$\dot{\gamma} = 1,\ \theta = 45^\circ.$$

Calculation of corresponding eigenvalues yields

$$\varkappa_1 = -0{,}05,\ \varkappa_2 = +0{,}025 + 0{,}446514i,$$
$$\varkappa_3 = +0{,}025 - 0{,}446514i.$$

Inserted in the integrated eqs. (117) these eigenvalues give the coordinates to the particle paths (since the coefficients c_{x1}, c_{z1}, c_{y2}, c_{y3} vanish, see eqs. (125)):

$$\begin{aligned} & x = e^{0{,}025t}\left(c_{x2}\cos(0{,}446514t) + c_{x3}\sin(0{,}446514t)\right). \\ (128)\quad & y = c_{y1}e^{-0{,}05t}, \\ & z = e^{0{,}025t}\left(c_{z2}\cos(0{,}446514t) + c_{z3}\sin(0{,}446514t)\right). \end{aligned}$$

The five coefficients are determined by the same method as in the example above, viz. from the following equations which are valid at $t=0$:

$$\begin{aligned} & (a)\quad x_o = c_{x2}, \\ & (b)\quad y_o = c_{y1}, \\ (129)\quad & (c)\quad z_o = c_{z2}, \\ & (d)\quad \dot{x} = a_{11}x_o + a_{13}z_o = -0{,}20x_o + 0{,}5z_o = \\ & \qquad = 0{,}025c_{x2} + 0{,}446514c_{x3}, \\ & (e)\quad \dot{z} = a_{31}x_o + a_{33}z_o = -0{,}5x_o + 0{,}25z_o = \\ & \qquad = 0{,}025c_{z2} + 0{,}446514c_{z3}. \end{aligned}$$

For explanation see eqs. (126).

With the thus determined coefficients introduced the particle-path equations read:

$$\begin{aligned} & (a)\quad x = e^{0{,}025t}\big(x_o\cos(0{,}446514t) + \\ & \quad + (1{,}119786z_o - 0{,}5039x_o)\sin(0{,}446514t)\big), \\ (130)\quad & (b)\quad y = y_o e^{-0{,}05t}, \\ & (c)\quad z = e^{0{,}025t}\big(z_o\cos(0{,}446514t) + \\ & \quad + (0{,}5039z_o - 1{,}119786z_o)\sin(0{,}446514t)\big). \end{aligned}$$

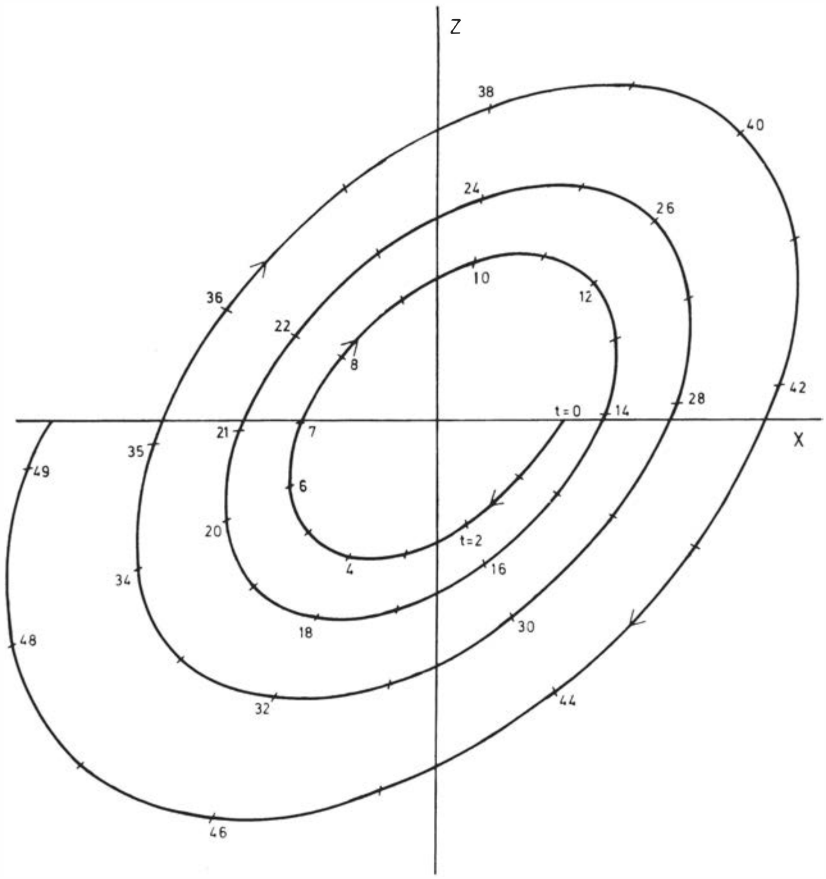


Fig. 18. Spiraling particle path in the plane $y = 0$ characteristic for a certain combination of 3-dimensional strains. Described by a particle initially at $x = 1$, $z = 0$. Numbers on path indicate positions at different time units after commencement of deformation. See text.

The only difference between this set of equations and those describing the movements in the previous example is in the sign of $\varkappa_1$ ($-0{,}05$) and the sign of the real part of $\varkappa_2$ and $\varkappa_3$, viz. $\alpha = 0{,}025$.

Also in this example the particle paths are spirals twisting around the y axis. However, now the particles move continuously away from the y axis at the same time as they circle about this axis. All particles, except those lying on the x, z plane, also exhibit a component of movement along the y and $-y$ axes, getting continuously closer to the x, z plane.

The movement trend in the x, z plane is shown in Fig. 18.

Progressive evolution of the strain ellipsoid. — We shall limit the discussion of the progressive evolution of the strain ellipsoid in three dimensions to the case with spiraling particle paths (i.e. complex eigenvalues of the rate-of-displacement matrix). In cases with real eigenvalues and hence simple-curve type particle paths the behavior of the strain ellipsoid differs only in degree — not in principle — from that already studied for the two contrasted orders of sequential superpositions. We may expect, however, that the analysis of the strain ellipsoid in cases with spiraling particle paths gives nontrivial and new information.

For the above special combination of irrotational three dimensional strain and simple shear that gave spiraling particle paths the equations for the paths (eqs. (127) p. 61) may be expressed in matrix form, thus:

$$(131)\quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & a_{33} \end{bmatrix} \begin{bmatrix} x_o \\ y_o \\ z_o \end{bmatrix}$$

in which

$$\begin{aligned} a_{11} &= \exp(\alpha t)\big(\cos(\beta t) - c_1 \sin(\beta t)\big) \\ a_{13} &= c_2 \exp(\alpha t)\sin(\beta t) \\ (132)\quad a_{22} &= \exp(\varkappa_1 t) \\ a_{31} &= -c_2 \exp(\alpha t)\sin(\beta t) \\ a_{33} &= \exp(\alpha t)\big(\cos(\beta t) + c_1 \sin(\beta t)\big) \end{aligned}$$

α is the real part and β the imaginary part of the two conjugate complex eigenvalues of the rate-of-displacement matrix and $\varkappa$ is the real eigenvalue. c_1 and c_2 are constants.

The expressions for x_o, y_o and z_o, which must be inserted in the equation for the initial sphere in order to obtain the equation for the strain ellipsoid, follow from inversion of the matrix above:

$$(133)\quad \begin{bmatrix} x_o \\ y_o \\ z_o \end{bmatrix} \begin{bmatrix} \frac{a_{33}}{\triangle'} & 0 & \frac{-a_{13}}{\triangle'} \\ 0 & a_{22}^{-1} & 0 \\ \frac{-a_{31}}{\triangle'} & 0 & \frac{a_{11}}{\triangle'} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Here $\triangle' = a_{11}a_{33} - a_{13}a_{31}$. A shorter notation for the same equations will be used in the development of the equation for the strain ellipsoid, viz.

$$(134)\quad \begin{bmatrix} x_o \\ y_o \\ z_o \end{bmatrix} = \begin{bmatrix} B_{11} & 0 & B_{13} \\ 0 & B_{22} & 0 \\ B_{31} & 0 & B_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Introduction of these formulas for x_o, y_o and z_o into the initial sphere with unit radius yields

$$(135)\quad (B_{11}^2 + B_{31}^2)x^2 + B_{22}^2 y^2 + (B_{13}^2 + B_{33}^2)z^2 + \\ + 2(B_{11}B_{13} + B_{31}B_{33})xz = 1,$$

which is the equation for the distorted sphere, i.e. the strain ellipsoid. To obtain the principal axes of the ellipsoid we follow the routine of transforming the equation to the matrix form and extract the eigenvalues.

$$(136)\quad (x\ y\ z) \begin{bmatrix} (B_{11}^2 + B_{31}^2) & 0 & (B_{11}B_{13} + B_{31}B_{33}) \\ 0 & B_{22}^2 & 0 \\ (B_{11}B_{13} + B_{31}B_{33}) & 0 & (B_{13}^2 + B_{33}^2) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

is the matrix form of the ellipsoid equation whose characteristic equation is

$$(137)\quad (B_{11}^2 + B_{31}^2 - \lambda)(B_{22}^2 - \lambda)(B_{13}^2 + B_{33}^2 - \lambda) - \\ - (B_{11}B_{13} + B_{31}B_{33})^2 (B_{22}^2 - \lambda) = 0.$$

One immediately sees that one root of this equation is

$$\lambda_1 = B_{22}^2.$$

To find the other two roots one divides by $(B_{22}^2 - \lambda)$ — note that $(B_{22}^2 - \lambda)$ is not zero when λ assumes values different from λ_1 — and obtain the second degree equation

$$(138)\quad (B_{11}^2 + B_{31}^2 - \lambda)(B_{13}^2 + B_{33}^2 - \lambda) - \\ - (B_{11}B_{13} + B_{31}B_{33})^2 = 0,$$

whose two roots are

$$(139)\quad \lambda i = \tfrac{1}{2}(B_{11}^2 + B_{33}^2 + B_{13}^2 + B_{31}^2) \pm \\ \pm \Big((B_{11}B_{13} + B_{31}B_{33})^2 - (B_{11}^2 + B_{31}^2)(B_{13}^2 + \\ + B_{33}^2) + \tfrac{1}{4}(B_{11}^2 + B_{13}^2 + B_{31}^2 + B_{33}^2)^2\Big)^{\frac{1}{2}}.$$

For any specified case with defined strain rates $\dot{\varepsilon}_x$, $\dot{\varepsilon}_y$, $\dot{\varepsilon}_z$ and $\dot{\gamma}$ and defined angle θ between the

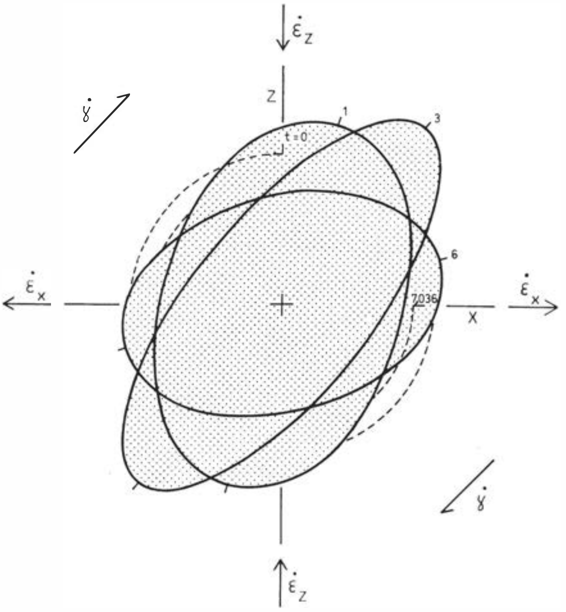


Fig. 19. The cross section in the x, z plane of the strain ellipsoid shown at $t = 1$, 3, 6 and 7,036 units of time after the commencement of progressive 3-dimensional strain of the kind also shown in Figs. 17 and 20. The outermost stipled circular curve outlines the size of the initial sphere from which the strain ellipsoid forms. The innermost stipled circular curve shows the size of the principal cross sections in the x, z plane of the strain ellipsoid at $t = 7,036$. For explanation see also text.

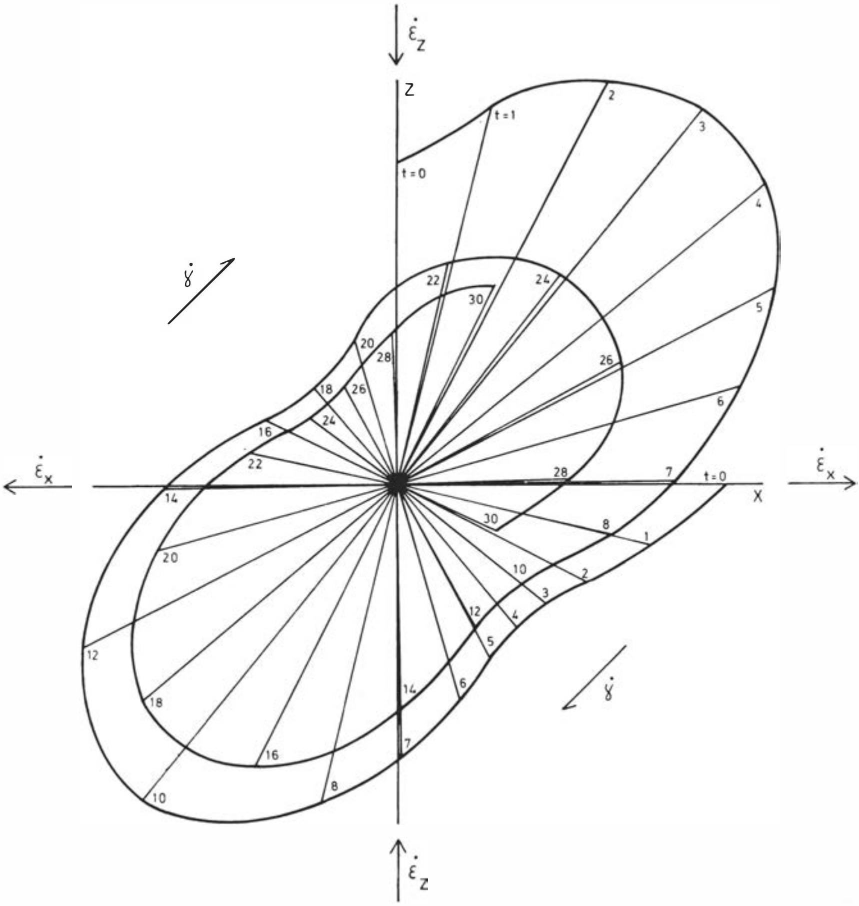


Fig. 20. Pattern showing orientation and length of two principal axes of strain ellipsoids formed during progressive composite 3-dimensional strain of the kind described in text. At $t = 0$ the strain commences and the starting points of the two spiraling curves cut off the same length of the x and the y axes. This length is the radius of the initial sphere which continuously deforms to the strain ellipsoid. The two spiraling curves starting at $t = 0$, one on the x axis, the other on the y axis, give the length and orientation of two of the principal strain axes at any time (up to 30 time units in the figure) after the straining has started. At $t = 3$, for example, the two principal axes coincide with the two radii which meet the two spirals at $t = 3$. These two radii are normal to one another — as is true with all pairs of radii with the same number — and one sees that the long principal axis lies in the first (and third) quadrant and the short principal axis lies in the fourth (and second) quadrant. At $t \approx 7$ units of time the principal axes coincide in direction with the axes developed at the very beginning of the deformation. The axes are now, however, shorter than initially, and both are equal in length. At $t \approx 7$ units of time the strain ellipsoid is hence bi-axial with the axis of rotational symmetry, which is the longer one, being parallel to the coordinate y axis.

x axis and the simple-shear direction, the elements a_{ij} in the matrix of eqs. (132) and consequently also the coefficients B_{ij} in eq. (139) are functions of t only. Consequently the roots of eq. (138) — i.e. the eigenvalues of the matrix of the strain-ellipsoid equation — and thus the lengths and orientations of the principal strain axes are functions of the time of evolution.

The numerical example 1 treated above has the following parameters needed for numerical calculation of the strain ellipsoid

$$\alpha = -0{,}025, \ \beta = 0{,}446514,$$
$$c_1 = 0{,}5039 \ \text{and} \ c_2 = 1{,}119786.$$

These values go via the elements a_{ij} of eq. (132) into the coefficients B_{ij} of formula (139) for the eigenvalues.

The eigenvalues and the length and orientation of the axes of the strain ellipsoid have been calculated for a sequence of times as shown in Figs. 19 and 20. We note that the long axes (parallel to y) increases continuously while the short and intermediate axes exhibit alternating shortening and lengthening at the same time as they rotate around the clock. To balance the continuous lengthening of the long axis the product of the intermediate and the short axes decreases continuously while they rotate and pulsate. Each time the short and intermediate axes return to and pass a given angular position they have become shortened. This is demonstrated in Fig. 20.

Appendix

In the discussion throughout this paper we have selected only examples in which neither the rates of strain, $\dot{\varepsilon}$ and $\dot{\gamma}$, nor the angular deviation, θ, between the simple shear direction and the principal axes of the irrotational part of the strain have varied with time and/or with position in space.

The reason for this limitation is not that constant strain rate is particularly applicable to natural rocks but rather that the elements of the matrix of the rate-of-displacement equations thereby become independent of time and space and we accordingly obtain a system of differential equations with constant coefficients. Such systems of differential equations are easy to integrate to generate the particle-path equations which constitutes the basis for the study of progressive deformation in systems undergoing strain such as deforming rocks. Our equations would be more applicable to conditions encountered in nature if they also would account for strain rates which vary with time and/or position in space.

Generally the integration of rate-of-displacement equation with variable coefficients — i.e. the generation of the corresponding particle-path equations — offers a difficult mathematical problem.

Under certain circumstances, however, systems of differential equations of the type examplified by the rate-of-displacement equations in this paper may readily by integrated even if the coefficients are not constant but are functions of time. Integration offers no special problem if each of the coefficients can be considered as a product of a constant and a time function, say $f(t)$, provided that the function $f(t)$ is the same for all coefficients. This condition allows us to consider some cases in which the rate of strain changes with time.

As a plane-strain example assume that the straining starts at constant rates $\dot{\varepsilon}_x$, $\dot{\varepsilon}_y$ and $\dot{\gamma}$ at time $t = 0$, but then increases with time as, say, the function $f(t) = \ln(e+t)$. This function is unity at $t = 0$ and increases firstly moderately fast with time and then more and more slowly.

Introduction of the factor $\ln(e+t)$ in, say, eq. (30) p. 40 yields

$$(140)\quad \begin{matrix}(a)\\(b)\end{matrix}\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}A\ln(e+t) & B\ln(e+t)\\ D\ln(e+t) & E\ln(e+t)\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix},$$

where we have used the notation:

$$(141)\quad \begin{aligned} A &= \dot{\varepsilon}_x - \dot{\gamma}\sin\theta\cos\theta\\ B &= \dot{\gamma}\cos^2\theta\\ D &= -\dot{\gamma}\sin^2\theta\\ E &= \dot{\varepsilon}_y + \dot{\gamma}\sin\theta\cos\theta. \end{aligned}$$

The solution of this system is

$$(142)\quad \begin{aligned} &(a)\quad x = c_{11}\exp[\varkappa_1\textstyle\int\ln(e+t)dt] + \\ &\qquad + c_{12}\exp[\varkappa_2\textstyle\int\ln(e+t)dt],\\ &(b)\quad y = c_{21}\exp[\varkappa_1\textstyle\int\ln(e+t)dt] + \\ &\qquad + c_{22}\exp[\varkappa_2\textstyle\int\ln(e+t)dt]. \end{aligned}$$

Here $\varkappa_1$ and $\varkappa_2$ are the eigenvalues of the matrix

$$\begin{bmatrix}\dot{\varepsilon}_x - \dot{\gamma}\sin\theta\cos\theta & \dot{\gamma}\cos^2\theta\\ -\dot{\gamma}\sin^2\theta & \dot{\varepsilon}_y + \dot{\gamma}\sin\theta\cos\theta\end{bmatrix} \equiv \begin{bmatrix}A & B\\ D & E\end{bmatrix},$$

and c_{ij} are coefficients partly determined by the initial coordinates of the point we wish to follow.

Since solutions of type (142) are not quite obvious and furthermore are not found in many books on applied mathematics we shall show that the solution is correct. If solution (142) is true then the differentiated form

$$(143)\quad \begin{aligned}\dot{x} &= c_{11}\varkappa_1\ln(e+t)\exp[\varkappa_1\textstyle\int\ln(e+t)dt] + \\ &+ c_{12}\varkappa_2\ln(e+t)\exp[\varkappa_2\textstyle\int\ln(e+t)dt]\end{aligned}$$

must be identical to (140 a):

$$(144)\quad \dot{x} = A\ln(e+t)x + B\ln(e+t)y.$$

It is now practical to rewrite eq. (143) thus

$$(145)\quad \begin{aligned} \dot{x} &= A\ln(e+t)\big(c_{11}\exp[\varkappa_1\textstyle\int\ln(e+t)dt] + \\ &+ c_{12}\exp[\varkappa_2\textstyle\int\ln(e+t)dt]\big) + \\ &+ (\varkappa_1 - A)\ln(e+t)c_{11}\exp[\varkappa_1\textstyle\int\ln(e+t)dt] + \\ &+ (\varkappa_2 - A)\ln(e+t)c_{12}\exp[\varkappa_2\textstyle\int\ln(e+t)dt], \end{aligned}$$

where the long term in the second parenthesis is the expression for x as given in equation 142 a. Consequently the expression

(146) $(\varkappa_1 - A)\ln(e+t)c_{11}\exp[\varkappa_1\int\ln(e+t)\mathrm{d}t] + (\varkappa_2 - A)\ln(e+t)c_{12}\exp[\varkappa_2\int\ln(e+t)\mathrm{d}t]$

in eq. (145) must be identically equal to

(147) $$B\ln(e+t)y,$$

or — with the expression for y from eq. (142 b) introduced — identical to

(148) $$B\ln(e+t)\left(c_{21}\exp[\varkappa_1\int\ln(e+t)\mathrm{d}t] + c_{22}\exp[\varkappa_2\int\ln(e+t)\mathrm{d}t]\right).$$

It follows that

(149) (a) $(\varkappa_1 - A)\ln(e+t)c_{11} = B\ln(e+t)c_{21}$ and

(b) $(\varkappa_2 - A)\ln(e+t)c_{12} = B\ln(e+t)c_{22}$.

Hence

(150) (a) $$c_{11}/c_{21} = \frac{B}{\varkappa_1 - A}$$ and

(b) $$c_{12}/c_{22} = \frac{B}{\varkappa_2 - A}.$$

That is the ratio between the coefficients c_{11} and c_{21} equals the ratio between the components of the first eigenvector of the matrix

$$\begin{bmatrix} A & B \\ D & E \end{bmatrix},$$

and c_{12} and c_{22} are related as the components of the second eigenvector to the same matrix. Note that the factor $\ln(e+t)$ in the matrix of the rate-of-displacement equations does not influence the ratio between the coefficients in the integrated equation.

The other two equations needed to determine completely the form of the coefficients c_{ij} follow by putting $x = x_o$ and $y = y_o$ at $t = 0$ in eqs. (142). Before this can be done, however, we must integrate the term $\ln(e+t)\mathrm{d}t$ in the exponent. This integral is $(t+e)\ln(e+t) - t$. Inserting this integral in eq. (142) and putting $x = x_o$ and $y = y_o$ at the same time as $t = 0$ yield

(151) (a) $$x_o = c_{11}e^{\varkappa_1 e} + c_{12}e^{\varkappa_2 e},$$

(b) $$y_o = c_{21}e^{\varkappa_1 e} + c_{22}e^{\varkappa_2 e}.$$

Equations (150) and (151) allow us to determine the four coefficients in the particle-path equations.

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