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COMPACTION OF SEDIMENTS

A Simple Mathematical Model for Calculating the Gravitational Porosity-Depth Equilibrium-Curve of Shales

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Abstract. Simple sets of difference equations are derived which allow the calculation of porosity-depth curves originating from the processes of gravitational non-elastic mechanical compaction, irrespective of possible effects caused by diagenetic changes. The application of the formulae yields the porosity-depth curve of Athy. Furthermore, there is some evidence that only the oldest layers of the young Tertiary shales of Venezuela and the Po valley (Italy), those belonging to the Hedberg type of porosity-depth curves, have as yet reached the state of compaction equilibrium, which implies that most parts of the sediments could still be compacting.

The compaction of sediments is a complex mechanism depending on many factors only partly known. Several important aspects are treated by J. M. Weller (1959), W. v. Engelhardt (1960, pp. 38–50), H. Füchtbauer & H.-E. Reineck (1963), W. v. Engelhardt (1967), H. Füchtbauer (1967), D. Heling (1967), the authors particularly quoted by J. M. Weller and other research workers. In this paper, the discussion will be restricted to the nonelastic compaction of clays and shales as a function of maximum depth of burial under the influence of their own weight (*autocompaction*) as well as under the influence of rocks burying the sediment after it has been deposited (*secondary weight compaction*). We are interested, therefore, in porosity changes due to permanent deformation, shrinking, or crushing of solid particles and their shifting into new positions etc., all of these phenomena being caused by an increase in petrostatic or hydrostatic pressure. On the other hand, compressibility, the effects of diagenetic changes such as the precipitation of minerals causing cementation, are neglected. Further, it is to be understood that the approach chosen is formal in the sense that only the simplest concepts of phenomenological physics are used, and that only states of equilibrium are considered. At every instant of time, the flow velocity of the interstitial water squeezed out by the compacting shale is assumed to correspond to the shrinking velocity of porosity; the squeezed-out water is allowed to escape from the porous system considered without hindrance so that an abnormal hydrostatic pressure can not arise. (Sediments with abnormal hydrostatic pressures are rarely encountered; the dynamics of compaction have been treated by Ortenblad (1929/30)). We assume that the change of porosity ε from the value ε_1 to the value ε_2 caused by an infinitesimal change of pressure p from p_1 to p_2 is proportional to ε_2,

$$(\varepsilon_1 - \varepsilon_2)/(p_1 - p_2) = -\varkappa\varepsilon_2,$$

the coefficient $\varkappa$ being the *coefficient of compaction*. For this equation to be useful, the meaning of its terms must be further explained. The derivations of formulae in the following constitute the refinement of a discussion of Engelhardt (1960, p. 45).

During the first stage of sedimentation, fresh mud is deposited with porosities ranging from approximately 35% and more to 80% and more. At the upper limit the water films adsorbed by sedimentary particles come into contact, while at the lower limit the water films have been squeezed out from between these particles, and contact is established between the mineral grains themselves (Weller, 1959, p. 282). At this stage, it will be reasonable to assume that the pressure p is approximately equal to the intergranular pressure, that Archimedes' principle of the hydraulic uplift of immersed bodies is valid and that $\varkappa$ is relatively large because the water films are not strong.

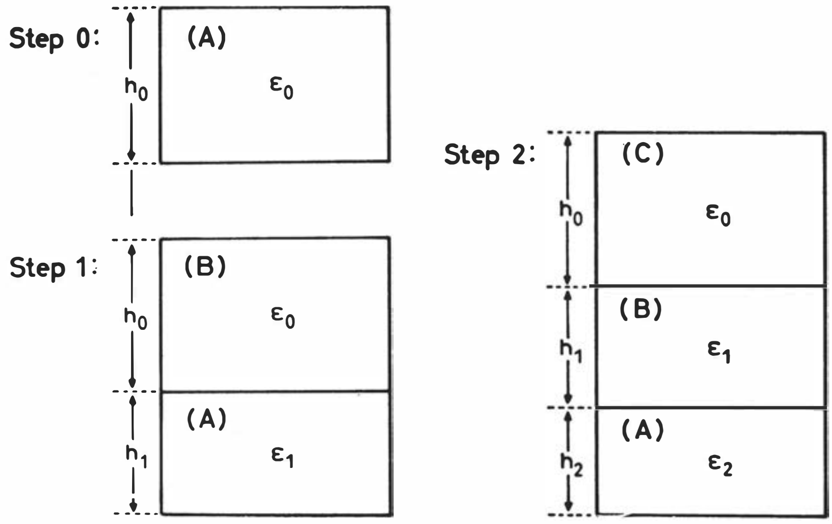

Fig. 1. Step by step deposition of porous strata.

The mud being further compacted down to porosities of approximately 10% or less, the "dewatering stage" gives way to a stage of mechanical deformation and dislocation, which should lead to the coefficient $\varkappa$ becoming smaller. In this second stage (as well as in the third one) the meaning of the symbol p becomes ambiguous. However, there are two limiting cases. Firstly, if the hydraulic uplift attains full effect, then the pressure p is equal to the petrostatic (intergranular) pressure, and there is an apparent loss of weight of the grains touching each other. Secondly, if as a consequence of the forces of adhesion and the smallness of the pores, Archimedes' principle is no longer effective, then the pressure p becomes proportional to the weight of the overlying column of solids and water.

During the third stage, beginning with porosities of 10% or less, the reduction of porosity increasingly requires the crushing and distortion of the harder components such as quartz (the clay minerals being already squeezed into the interstices, (cf. Weller, 1959, p. 283), resulting in a further decrease of $\varkappa$. These remarks show that $\varkappa$ above all should be a function of the porosity ε, differing somewhat according to the mineralogical composition and the shape and size distribution of the clay.

Now we are in a position to determine porosity curves by examining the deposition of porous strata within a through step by step. Firstly, we consider the limiting case with full effectiveness of the hydraulic uplift. In this case, p as function of depth of burial H is the petrostatic pressure. On the average, p is equal to the apparent weight of the matrix of solids belonging to a column of sediment with cross section 1 divided by $1-\varepsilon$, the cross section of the solid matrix at depth H, the number ε (measured as fraction of 1) being the porosity at depth H. With H increasing, the weight of the matrix as well as its cross section $1-\varepsilon$ will increase.

The steps (cf. Fig. 1):

Step 0: A layer (A) of fresh mud of thickness h_0, area F, apparent weight M of solid particles and porosity ε_0, is deposited.

Step 1: A layer (B) of fresh mud of thickness h_0, area F, apparent weight of solid particles M and porosity ε_0, is deposited and buries layer (A). Under the influence of the weight of layer (B) the porosity of layer (A) is reduced to ε_1, with a corresponding reduction in thickness from h_0 to h_1. At the surface of layer (A), the apparent weight M of solid particles of layer (B) is applied to a surface of solids $(1-\varepsilon_0)F$, changing the *average* petrostatic pressure from zero to $M/(1-\varepsilon_0)F$. Hence

$$(\varepsilon_1-\varepsilon_0)\Big/\left\{\frac{M}{(1-\varepsilon_0)F}-0\right\} = -\varkappa_1\varepsilon_1,$$

$\varkappa_1$ being the coefficient of compaction of ε_1. (The atmospheric pressure as compared therewith is so small that it can be ignored.)

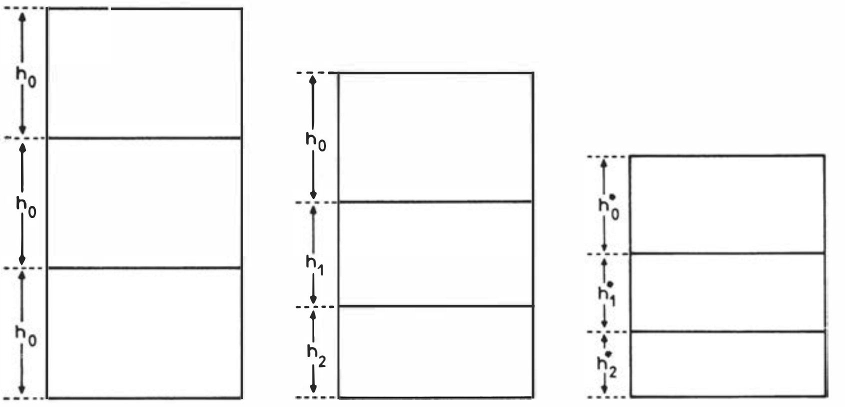

Fig. 2. Step by step deposition of porous strata. (*Left*) Three uncompacted strata of thickness h_0 each. (*Center*) As a consequence of autocompaction, the middle stratum is reduced to thickness h_1, the lower stratum to h_2. (*Right*) As a consequence of an additional overburden, the thickness of the strata is reduced further.

Step 2: A layer (C) of fresh mud of thickness h_0, area F, apparent weight of solid particles M and porosity ε_0, is deposited and buries layer (B). Under the influence of the weight of layer (C) the porosity of layer (B) is reduced to ε_1, with a corresponding reduction in thickness from h_0 to h_1. At the surface of layer (A), the petrostatic pressure changes from $M/(1-\varepsilon_1)F$ to $2M/(1-\varepsilon_1)F$, reducing ε_1 to ε_2 and h_1 to h_2:

$$(\varepsilon_2-\varepsilon_1)\Big/\left\{\frac{2M}{(1-\varepsilon_1)F}-\frac{M}{(1-\varepsilon_1)F}\right\}=-\varkappa_2\varepsilon_2$$

Step i: Layer (A) is compacted according to the formula

$$(\varepsilon_i-\varepsilon_{i-1})\Big/\left\{\frac{iM}{(1-\varepsilon_{i-1})F}-\frac{(i-1)M}{(1-\varepsilon_{i-1})F}\right\}=-\varkappa_i\varepsilon_i,$$

resulting in the *general autocompaction formula*

$$\varepsilon_i=\varepsilon_{i-1}(1-\varepsilon_{i-1})/(1-\varepsilon_{i-1}+\varkappa_i m)\quad i=1,2,3,\dots \qquad (1),$$

the constant $m=M/F$ being the apparent weight of the solids of any layer per unit area. Using the CGS-system of units and the densities 2.7 and 1.1 [g/cm^3] for solid particles and the interstitial brine resp., the value

$$m=981(2.7-1.1)(1-\varepsilon_0)h_0 \qquad (2)$$

will be obtained. (The value 981 [cm/sec^2] is the acceleration due to gravity.) The porosity ε_i is encountered at the depth of burial H_i where

$$H_i=h_0+h_1+h_2+\dots+h_i \qquad (3)$$

and

$$h_i=(1-\varepsilon_0)h_0/(1-\varepsilon_i). \qquad (4)$$

The last equation is a consequence of the conservation of apparent mass of the solid particles:

$$m=1.6\times981(1-\varepsilon_0)h_0=1.6\times981(1-\varepsilon_1)h_1$$
$$=1.6\times981(1-\varepsilon_2)h_2=\dots$$

It should be noted that the solution of the system (1)–(4) as well as the solution of the following systems need not begin with the initial values $\varepsilon=\varepsilon_0$, $H_0=h_0$. The calculations may begin with any value of ε and its corresponding depth H_ε. In this case, $H_i=H_\varepsilon+h_1+h_2+\dots+h_i$, ε_0 being the chosen starting value of ε.

From the point of view of numerical mathematics, the system of difference equations (1)–(4) is rather crude, but it has the advantage that its correspondence to the actual deposition of mud is easy to grasp. The accuracy of the result will be sufficient if the arbitrary initial thickness h_0 is not chosen too large.

When the deposition of clay has ceased, the surface of the sediment may be covered by rock of total weight W, corresponding to a weight $w=W/F$ per unit area surface, resulting in a further reduction of porosity from ε_i to ε_i^* and in a reduction of thickness from h_i to to h_i^* (Fig. 2). This process may be formally described by the equation

$$(\varepsilon_i^*-\varepsilon_i)\Big/\left\{\frac{im+w}{1-\varepsilon_i}-\frac{im}{1-\varepsilon_i}\right\}=-\varkappa_i^*\varepsilon_i^*,$$

$\varkappa_i^*$ being the coefficient of compaction belonging to ε_i^*. Hence

$$\varepsilon_i^* = \varepsilon_i(1-\varepsilon_i)/(1-\varepsilon_i+\varkappa_i^* w) \tag{1*}$$

and

$$H_i^* = h_0^* + h_1^* + \cdots + h_i^* \tag{3*}$$

where

$$h_i^* = (1-\varepsilon_0)\,h_0/(1-\varepsilon_i^*) \tag{4*}$$

Up to now, all formulae have been derived by using an *average* petrostatic pressure rise $m/(1-\varepsilon_{i-1})$ and $w/(1-\varepsilon_i)$ respectively, resulting in the graphs of $\varepsilon = \varepsilon(H)$ and $\varepsilon^* = \varepsilon^*(H^*)$ as smooth curves. However, the actual petrostatic pressure will differ from point to point within each horizontal plane of equal depth according to the distribution function of the planes of contact of different grains. This effect may be estimated easily by using the same shape of distribution curve for p with mean $m/(1-\varepsilon_{i-1})$ and $w/(1-\varepsilon_i)$ respectively and applying the Monte Carlo methods to the above formulae. The result will be a set of zig-zag-curves mainly located within a strip the breadth of which depends on the variance of the distribution function of p. Obviously, it would be easy, to incorporate a statistical or regular variation of the densities of the solid particles and the interstitial solutions.

Let us now consider the second limiting case, assuming that Archimedes' principle is no longer operative and that, consequently, the pressure p on some horizontal cross section of area 1 is numerically equal to the total weight of the overlying column of solid particles and interstitial solutions.

Step 0: A layer (A) of fresh mud of thickness h_0', area F, porosity ε_0' and total weight $\{1.1\varepsilon_0' + 2.7(1-\varepsilon_0')\}gh_0'F$, is deposited, 1.1 being the density of the interstitial brine, 2.7 the density of the solid particles and $g = 981$ [cm/sec²].

Step 1: A layer (B) of fresh mud of thickness h_0', area F and porosity ε_0', is deposited and buries layer (A). Under the influence of the weight of layer (B) the porosity of layer (A) is reduced to ε_0' with a corresponding reduction in thickness from h_0' to h_1'. At the surface of layer (A), the total weight $\{1.1\,\varepsilon_0' + 2.7(1-\varepsilon_0')\}gh_0'F$ of layer (B) takes effect, changing the average total pressure from zero to $\{1.1\varepsilon_0' + 2.7(1-\varepsilon_0')\}gh_0'$. Hence

$$(\varepsilon_1' - \varepsilon_0')/\{1.1\varepsilon_0' + 2.7(1-\varepsilon_0')\}gh_0' = -\varkappa_1'\varepsilon_1',$$

$\varkappa_1'$ being the coefficient of compaction belonging to ε_1'.

Step 2: A layer (C) of fresh mud of thickness h_0', area F and porosity ε_0' is deposited and buries layer (B). Under the influence of the weight of layer (C) the porosity of layer (B) is reduced to ε_1' with a corresponding reduction in thickness from h_0' to h_1'. At the surface of layer (A), the total weight increases from $\{1.1\varepsilon_0' + 2.7(1-\varepsilon_0')\}gh_0'$ to

$$\{1.1\varepsilon_1' + 2.7(1-\varepsilon_1')\}gh_1' + \{1.1\varepsilon_0' + 2.7(1-\varepsilon_0')\}gh_0'.$$

Consequently

$$(\varepsilon_2' - \varepsilon_1')/\{1.1\,\varepsilon_1' + 2.7(1-\varepsilon_1')\}gh_1' = -\varkappa_2'\varepsilon_2'.$$

Step i: Layer (A) is compacted according to the formula

$$(\varepsilon_i' - \varepsilon_{i-1}')/\{1.1\,\varepsilon_{i-1}' + 2.7(1-\varepsilon_{i-1}')\}gh_{i-1}' = -\varkappa_i'\varepsilon_i',$$

yielding the *general autocompaction formula*

$$\varepsilon_i' = \varepsilon_{i-1}'/\{1 + (1-\varepsilon_0')\,h_0'g\psi_{i-1}\varkappa_i'\}$$

where

$$\psi_{i-1} = 1.6 + \{1.1/(1-\varepsilon_{i-1}')\}, \quad i = 1, 2, 3, \dots \tag{1'}$$

because of

$$h_{i-1}' = (1-\varepsilon')\,h_0'/(1-\varepsilon_{i-1}');$$

cf. equation (4).

If the sediment is covered by a rock applying a pressure w to the surface of the shale, then the porosity ε_i' is reduced to the value ε_i'' according to the formula $(\varepsilon_i'' - \varepsilon_i')/w = -\varkappa_i''\varepsilon_i''$, resulting in

$$\varepsilon_i'' = \varepsilon_i'/(1 + \varkappa_i'' w). \tag{1''}$$

If the coefficient of compaction $\varkappa$ is known as a function of porosity ε, then the corresponding porosity-depth curves can be calculated by means of the difference equations, or, if a porosity-depth curve is given, then $\varkappa$ may be estimated. According to Terzaghi (1925) and v. Engelhardt (1960, p. 44) the empirical relation

$$\frac{\varepsilon}{1-\varepsilon} = \frac{\varepsilon_0}{1-\varepsilon_0} - b \cdot \log p$$

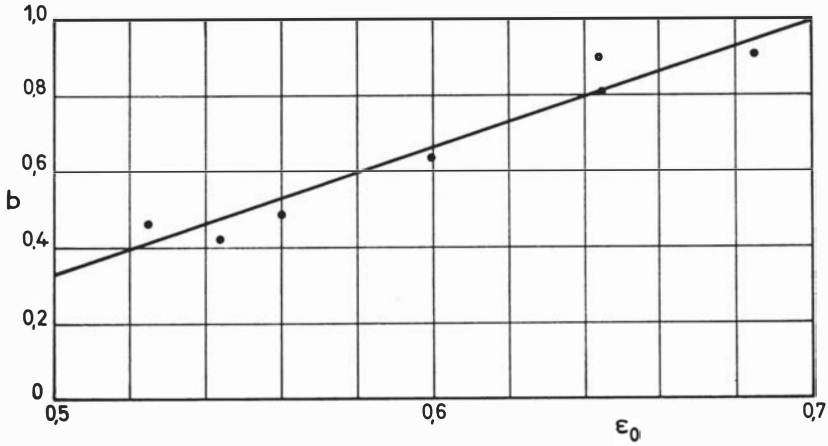

Fig. 3. The dependence of Terzaghi's parameter b on the initial porosity ε_0, according to measurements of various authors, cf. v. Engelhardt (1960, p. 44).

is valid for pressures at least up to approximately 20 atms, if the pressure p is applied to a thin slice of clay in such a way that the squeezed-out interstitial water is allowed to escape at once. ε_0 is the porosity at atmospheric pressure, and in the range $0.5 \leqslant \varepsilon_0 \leqslant 0.7$ the parameter b is roughly a linear function of ε_0 (cf. Fig. 3). Solving the Terzaghi equation for p yields

$$p = \exp\left\{\left(\frac{\varepsilon_0}{1-\varepsilon_0} - \frac{\varepsilon}{1-\varepsilon}\right)\Big/ 0.4343 \times b\right\} \quad [\text{atm}].$$

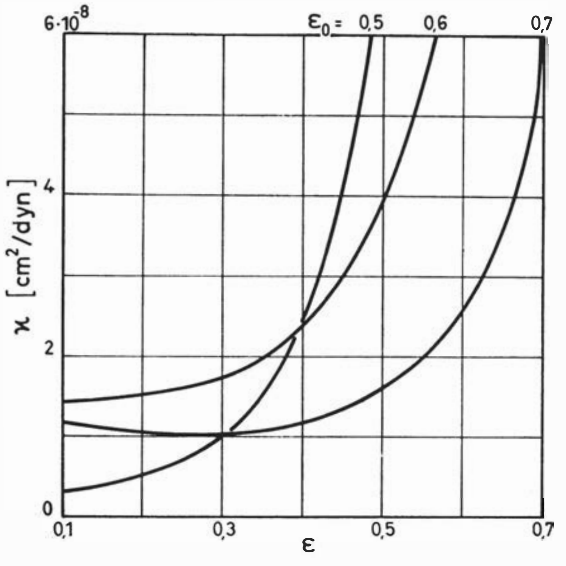

Fig. 4. The dependence of the coefficient of compaction $\varkappa$ on the porosity ε and the initial porosity ε_0, calculated by means of Terzaghi's equation.

In the simple case where a pressure p is applied to a thin slice of clay, the equation $(\varepsilon_1 - \varepsilon_2)/(p_1 - p_2) = -\varkappa\varepsilon_2$ may be written in the form $d\varepsilon/dp = -\varkappa\varepsilon$, yielding

$$\varkappa = \frac{ln\,(\varepsilon_0/\varepsilon)}{p-1}$$

after integration (p, [atm]; $\varkappa$, [atm^{-1}]; 1 atm = 981000 dyn/cm^2), and insertion of p yields $\varkappa$ as function of ε (cf. Fig. 4).

These values of $\varkappa$ may be compared to values of $\varkappa$ calculated from known porosity-depth curves by means of the difference equations, *assuming that the observed variation of porosity is mainly a result of gravitational processes.* Choosing the values of Hedberg from Venezuela (Hedberg, 1936, p. 254) and the curve of Athy from Oklahoma (Athy, 1930, p. 13) shown in Fig. 5, the graphs of Fig. 6 are obtained.

As Fig. 4 indicates, the coefficient of compaction decreases rapidly during the dewatering stage, becoming more or less stabilized at a value of $\varkappa = 10^{-8}$ [cm^2/dyn] as soon as the stage of mechanical deformation and dislocation of the clay particles is reached. As Figs. 6 and 7 show, the value $\varkappa = 10^{-8}$ is in good agreement with the Athy curve if Archimedes' principle is assumed to be effective and if, consequently, the compaction is caused by the intergranular pressure only. As soon as the crushing stage is reached, the coefficient of compaction decreases rapidly, as to be expected (cf. Fig. 6).

On the other hand, the $\varkappa$-curves for Tertiary sedi-

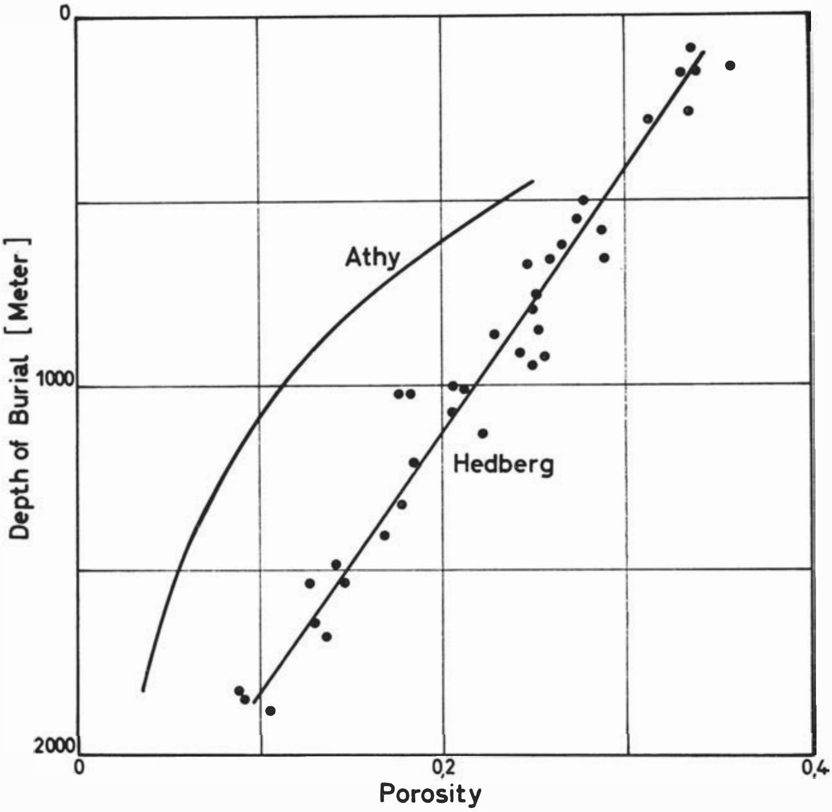

Fig. 5. The porosity-depth curves of Athy and Hedberg. The latter largely corresponds with curves for the Po valley (Italy), whereas the Athy curve roughly corresponds to Fücht-bauer's curve for the Lias of Northwestern Germany, cf. v. Engelhardt (1960, p. 41) and Roll (1956, p. 236).

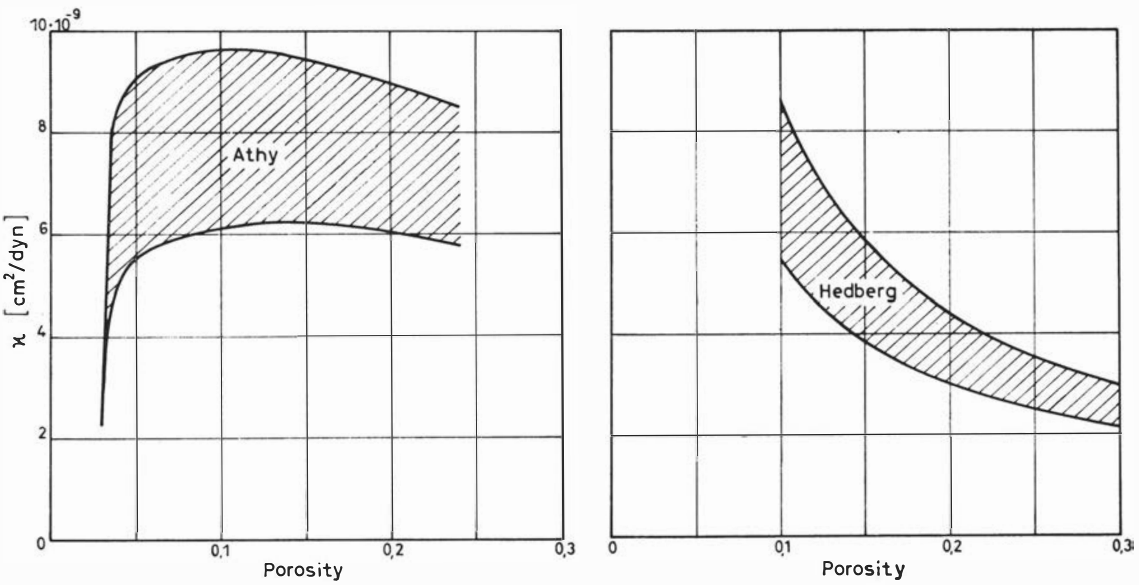

Fig. 6. The dependence of the coefficient of compaction $\varkappa$ on porosity calculated from the curves of Athy and Hedberg. The upper-limit curves refer to the case of full activity of hydraulic uplift, the lower-limit curves refer to the case of no uplift.

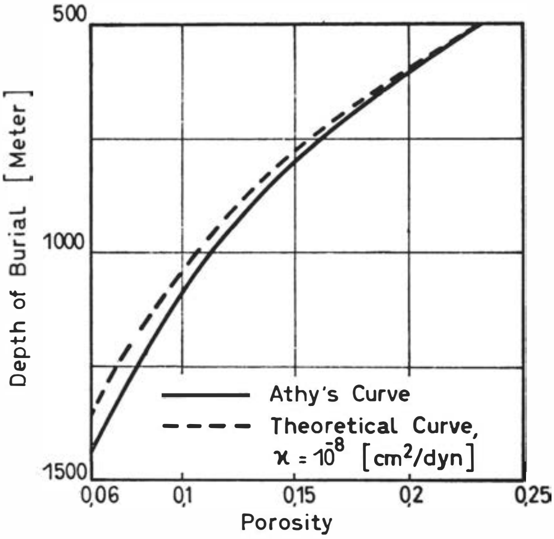

Fig. 7. Comparison of Athy's curve with a theoretical one using a constant value of $\varkappa$ and the system (1)–(4) and $\varepsilon = 0.25$ as the initial value at a depth of 500 m.

ments of Venezuela and Italy (Hedberg type) show that the value $\varkappa = 10^{-8}$ is hardly reached even in those layers buried deepest, whereas the younger

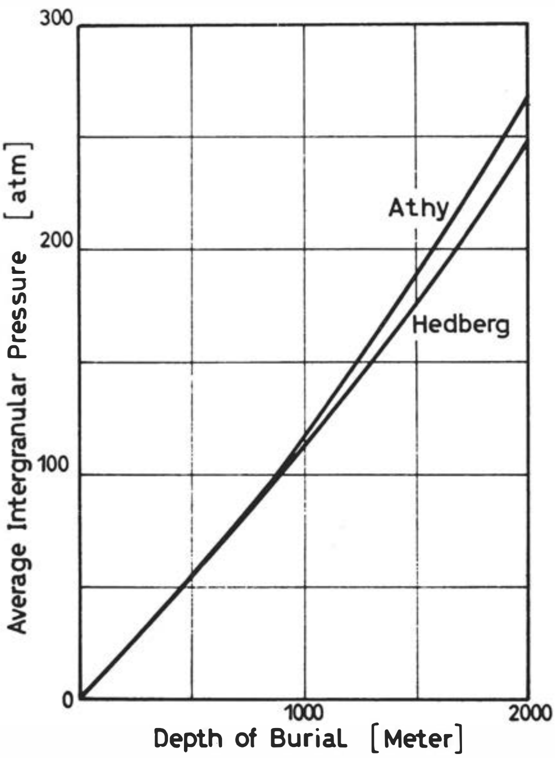

Fig. 8. The variation of intergranular pressure with depth of burial.

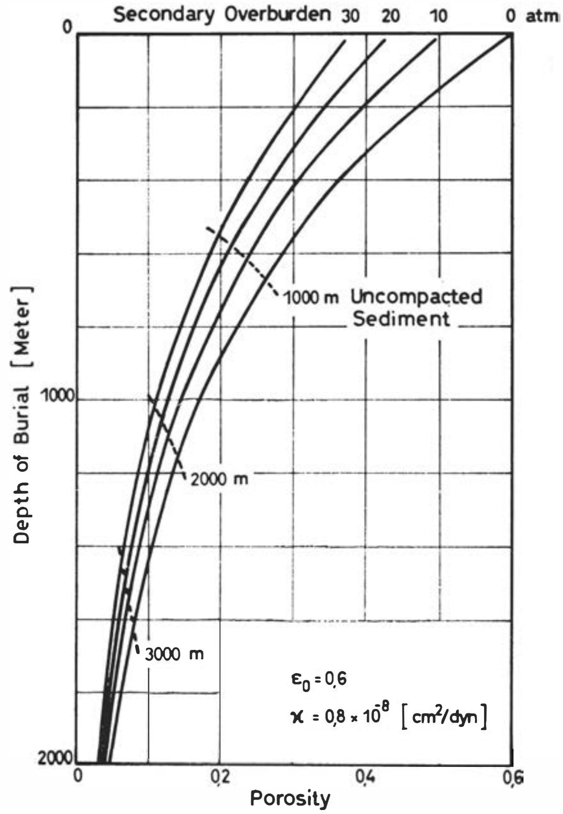

Fig. 9. The effect of secondary overburden on the porosity-depth curve. An additional pressure of 30 atm is roughly equivalent to the pressure exerted by a rock with negligible porosity and a thickness of 100 m.

ones show a gradual decrease in $\varkappa$ down to one fourth of 10^{-8}. This could indicate that the Hedberg type constitutes a non-equilibrium stage of compaction in which the younger layers are further from the equilibrium position than the older ones. If this be true, it could imply that shales of the Hedberg type are still in the process of compaction.

Finally, we wish to show the increase of petrostatic pressure p with depth of burial H (Fig. 8), and the influence of secondary overburden (Fig. 9), calculated by means of the system (1*), (3*), (4*). If ε is the porosity at depth H [cm], $\varepsilon(z)$ the porosity at depth z [cm], and $\bar{\varepsilon}$ the average porosity between the limits 0 and H, then the formula

$$p - \frac{981(2.7-1.1)}{1-\varepsilon}\int_0^H \{1-\varepsilon(z)\}\,dz$$

$$= 1569.6 \times H \times \frac{1-\bar{\varepsilon}}{1-\varepsilon}\,[\text{dyn/cm}^2]$$

is valid.

ACKNOWLEDGEMENT

This paper is dedicated to Dr. Dr. h.c. A. Roll, who, as the director of the Geological Division of Elwerath, has done so much to encourage basic research and to whom the writers are deeply indebted.

Sommaire. Dans cet ouvrage, on se sert de simples équations de différence pour établir des courbes de porosité et de profondeur, provenant des processus de gravitation. L'application des formules mène à la courbe d'Athy. En outre, on montre que la courbe type Hedberg, qui caractérise les jeunes argiles tertiaires de la vallée du Po et du Venezuela, peut être expliquée par des déséquilibres et qu'en relation avec ces faits, la compacité des jeunes sédiments pourraient encore aujourd'hui augmenter.

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